Criteria of Renormalizability in Effective Field Theories

A. M. Gasparyan, Ruhr-Universität Bochum

in collaboration with E. Epelbaum, N. Jacobi

August 26, 2024, Chiral Dynamics 2024, Bochum

Renormalization in nuclear chiral EFT

Expansion parameter: (soft scale)/(hard scale) $Q = \dfrac{q}{\Lambda_b}$

$$q \in \{|\vec{p}|\,, M_\pi\}\,, \qquad \Lambda_b \sim M_\rho$$

"Perturbative" calculation of observables

Renormalization in nuclear chiral EFT

Expansion parameter: (soft scale)/(hard scale) $Q = \dfrac{q}{\Lambda_b}$

$$q \in \{ |\vec{p}| \, , M_\pi \} \, , \qquad \Lambda_b \sim M_\rho$$

“Perturbative” calculation of observables

$$\mathcal{L}_{\rm eff} = \mathcal{L}_\pi^{(2)} + \mathcal{L}_{\pi N}^{(1)} + \mathcal{L}_{NN}^{(0)} + \mathcal{L}_{NN}^{(2)} + \ldots$$

Bare parameters of the Lagrangian → Renormalization: power counting for renormalized quantities

Implicit renormalization

$T = T_0$ fit bare $C_i^{(0)}$

$T = T_0 + T_2$ (re)fit bare $C_i^{(0)},\ C_i^{(2)}$

$T = T_0 + T_2 + T_4$ (re)fit bare $C_i^{(0)},\ C_i^{(2)},\ C_i^{(4)}$

$\ldots$

Balancing at the border of phenomenology

Implicit renormalization

$T = T_0$ fit bare $C_i^{(0)}$

$T = T_0 + T_2$ (re)fit bare $C_i^{(0)}, C_i^{(2)}$

$T = T_0 + T_2 + T_4$ (re)fit bare $C_i^{(0)}, C_i^{(2)}, C_i^{(4)}$

$\ldots$

Balancing at the border of phenomenology

Explicit renormalization

$$T = \mathbb{R}(T_0) + \mathbb{R}(T_2) + \mathbb{R}(T_4) + \ldots$$

$C_i = C_i^r + \delta C_i$ bare =renormalized + counter term
(absorb divergent and power counting breaking contributions)

Identify each term individually, or at least prove this is possible

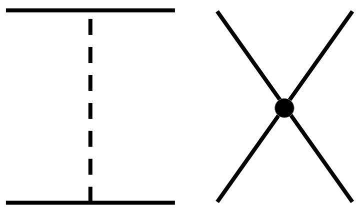
Justifies theoretical error estimation!

Power counting for NN chiral EFT: LO and NLO

Weinberg, S., **NPB363**, 3 (1991)

$\mathcal{O}(Q^0)$

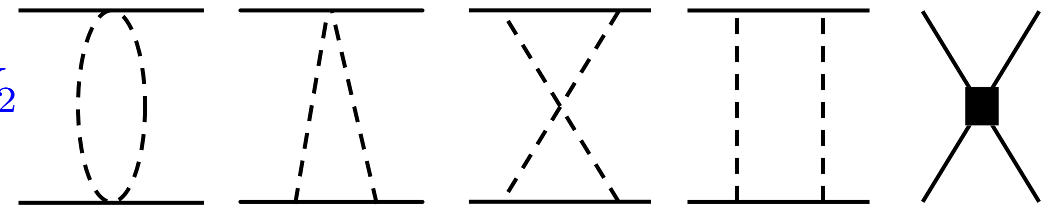
$V_{\mathrm{LO}} = V_0$



+...

$\mathcal{O}(Q^2)$

$V_{\mathrm{NLO}} = V_2$

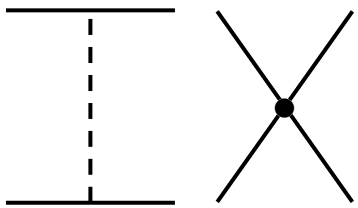


Power counting for NN chiral EFT: LO and NLO

Weinberg, S., **NPB363**, 3 (1991)

$\mathcal{O}(Q^0)$

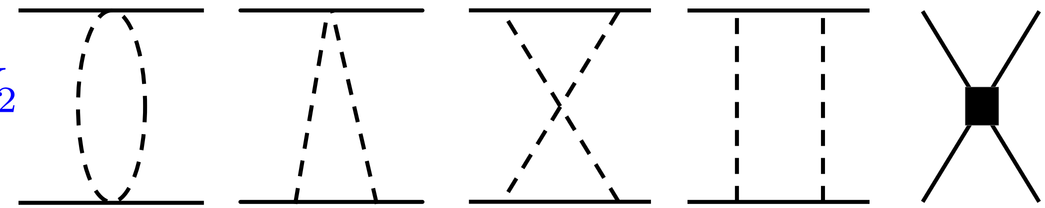
$V_{\text{LO}} = V_0$



+...

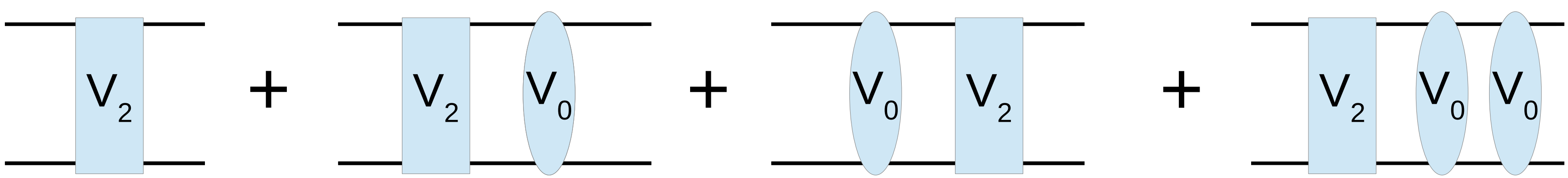
$\mathcal{O}(Q^2)$

$V_{\text{NLO}} = V_2$



LO potential has to be iterated (resummed):

$$T_2 = V_2 + V_2 G V_0 + V_0 G V_2 + V_2 G V_0 G V_0 + \ldots$$



Regulator:
finite (of the order of the breakdowen scale)
cutoff scheme

LO: $T_0^{[n]} = V_0 (G V_0)^n \sim \mathcal{O}(Q^0)$ Infinite number of divergencies of different order

We cannot absorb all positive powers of Λ by counter terms.
Absorb only those that are not compensated by the inverse powers of Λ_b

Expectation: $\Lambda \approx \Lambda_b : \int \frac{p^{n-1} dp}{(\Lambda_b)^n} \sim \left(\frac{\Lambda}{\Lambda_b}\right)^n \sim \left(\frac{\Lambda_b}{\Lambda_b}\right)^n \sim \mathcal{O}(Q^0)$

G. P. Lepage, nucl-th/9706029
J. Gegelia, **JPG25**, 1681 (1999)

NLO: $T_2^{[m,n]} = (V_0 G)^m \, V_2 \, (G V_0)^n \sim \mathcal{O}(Q^0) \neq \mathcal{O}(Q^2)$

Power-counting violating contributions from large loop momenta: $p \sim \Lambda \,, p' \sim \Lambda$ in $V_2(p', p)$

Expectation: power-counting breaking contributions
can be absorbed by lower order contact (counter) terms

Effective Lagrangian and the regulator as the renormalization scheme

$$\boxed{\mathcal{L} = \mathcal{L}_{\mathrm{LO}} + \mathcal{L}_{>\mathrm{LO}}}$$

$\mathcal{L}_{\mathrm{LO}} \Longrightarrow V_{\mathrm{LO}} = V_{\Lambda}^{(0)}$ $\longrightarrow$ regulated potential

$V_{>\mathrm{LO}} = V - V_{\mathrm{LO}}$ contains $\delta_{\Lambda} V^{(0)} = V_{\Lambda}^{(0)} - V_{\Lambda=\infty}^{(0)}$

Lagrangian and amplitude are **formally** cutoff (regulator) independent

$\dfrac{d\mathcal{L}}{d\Lambda} = 0$ $\longrightarrow$ $\dfrac{dT}{d\Lambda} = 0$ -RG-invariance

Λ specifies the non-perturbative regime (the number of bound states)

Λ plays a role of the renormalization scale

The remaining Λ-dependence is removed perturbatively by expansion in δV_{Λ}

For locally regulated long-range potentials, $\delta_{\Lambda} V^{(0)}$ can be expanded in 1/Λ
and absorbed by contact interactions,
or can be kept explicit to access lower values of the cutoff

Two approaches to renormalization

Finite cutoff approach: find the cutoff window where renormalization and perturbative expansion work with the minimal number of contact interactions: naive dimensional analysis + possible promotions

Analogous to choosing the optimal renormalization scale μ in QED or QCD (Brodsky, Lepage, Mackenzie, **PRD28 (1)**, 228 (1983))

Practically efficient and consistent with principles of EFTs

"RG-invariant" scheme: varying the cutoff $\Lambda_b<\Lambda<\infty$, determine the necessary number of the contact interactions by means of numerical checks

H. Grießhammer, **EPJA 56 (4)**, 118 (2020)

H. W. Hammer, S. König, U. van Kolck, **Rev. Mod. Phys. 92(2)**, 025004 (2020)

$$\frac{\Lambda}{T^{(\mathcal{V})}(Q,\Lambda)}\frac{\mathrm{d}T^{(\mathcal{V})}(Q,\Lambda)}{\mathrm{d}\Lambda}=\mathcal{O}\left(\frac{Q^{\mathcal{V}+1}}{M_{\mathrm{hi}}^{\mathcal{V}}\Lambda}\right)$$

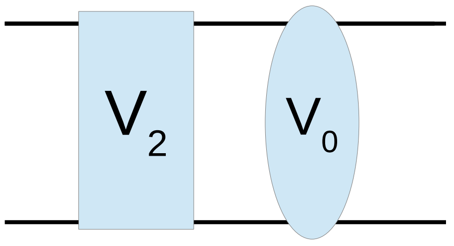
-existence of a $\Lambda\to\infty$ limit

This requirement seemingly cannot be fulfilled (exceptional cutoffs)

AG, E.Epelbaum, **PRC107**, 034001 (2023)

R. Peng, B. Long, F. Xu, 2407.08342 (2024)

Technicalities of renormalization: estimating integrals using bounds on potentials



$$T_2^{[0,1]} = \int_0^\infty \frac{p''^2 dp''}{(2\pi)^3} V_2(p', p'') G(p''; p_{\rm on}) V_0(p'', p)$$

LO potential: $V_0 \sim 1$

NLO potential: $V_2 \sim \frac{p^2}{\Lambda_b^2}\left(\log \frac{p^2}{M_\pi^2} + 1\right)$

2-nucleon Green's function: $G \sim \frac{1}{p^2}$

Integral converges at $p \sim \Lambda$ (regulator) $\longrightarrow$ $T_2^{[0,1]} \sim \frac{\Lambda^3}{\Lambda_b^3} \log \frac{\Lambda}{M_\pi} \neq \mathcal{O}(Q^2)$

Renormalization → Subtraction → Counter term δC_0

Structure of the interaction in chiral EFT

Interaction obtained from chiral EFT: $V(\vec{p}\,', \vec{p}) = V_{\rm short}(\vec{p}\,', \vec{p}) + V_{\rm long}(\vec{p}\,', \vec{p})$

$$V_{\rm short}(\vec{p}\,', \vec{p}) = \text{Polynomial}(\vec{p}\,', \vec{p})\, F_\Lambda(\vec{p}\,', \vec{p})$$

$$V_{\rm long}(\vec{p}\,', \vec{p}) = V_L(\vec{q} = \vec{p}\,' - \vec{p})\, \tilde{F}_\Lambda(\vec{p}\,', \vec{p}), \qquad V_L = V_{1\pi} + V_{2\pi} + \ldots$$

Subtractions: $\left| V(p\,', p) - V(p\,', 0) \right| \le \left| \frac{p}{p'} \right| \times (\ldots) \text{ if } |p'| > |p|$

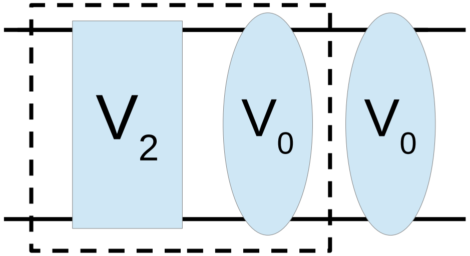
$$\left| V(p\,', p) - \sum_{i=0}^{n} \frac{\partial^i V(p', p)}{i! (\partial p)^i} \bigg|_{p=0} p^i \right| \le \left| \frac{p}{p'} \right|^{n+1} \times (\ldots) \text{ if } |p'| > |p|$$

AG, E.Epelbaum, **PRC 105**, 024001 (2022)

→ Large loop momenta are suppressed

→ Renormalizability

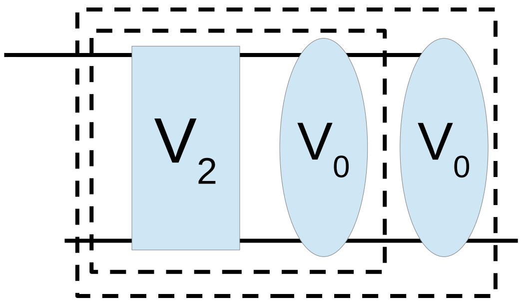
Iterations of V_0



Renormalized subdiagram: $\mathbb{R}\left(T_2^{[0,1]}\right) \sim \frac{p^2}{\Lambda_b^2}\frac{\Lambda}{\Lambda_b}\log\frac{\Lambda}{M_\pi} = \mathcal{O}(Q^2)$

Integral converges at $p \sim \Lambda$ $\longrightarrow$ $T_2^{[0,2]} \sim \frac{\Lambda^4}{\Lambda_b^4}\log\frac{\Lambda}{M_\pi} \neq \mathcal{O}(Q^2)$

One more subtraction: the same form of a counter term δC_0



$$\mathbb{R}\left(T_2^{[0,2]}\right) \sim \frac{p^2}{\Lambda_b^2}\frac{\Lambda^2}{\Lambda_b^2}\log\frac{\Lambda}{M_\pi}$$

General case: $\mathbb{R}\left(T_2\right) = \sum_{m,n=0}^{\infty} \mathbb{R}\left(T_2^{[m,n]}\right)$ -Perturbative (convergent) sum

$$\boxed{\mathbb{R}(T_2^{[m,n]})(p) \sim \frac{p^2}{\Lambda_b^2}\left(\frac{\Lambda}{\Lambda_b}\right)^{m+n}\log\Lambda/M_\pi = \mathcal{O}(Q^2)}$$

AG, E.Epelbaum, **PRC 105**, 024001 (2022)

$\Lambda \approx \Lambda_b$

Counterexample:
Non-local separable long-range interaction

AG, E.Epelbaum, N.Jacobi, in preparation

$$V_0 = C_0 F_\Lambda(p') F_\Lambda(p) + \ldots,$$

$$V_2 = C_2 \frac{p'^2 + p^2}{\Lambda_b^2} \frac{p'^2 p^2}{(M_\pi^2 + p'^2)(M_\pi^2 + p^2)} F_\Lambda(p') F_\Lambda(p).$$ two-pion exchange

$$F_\Lambda(p) = \frac{\Lambda^2}{(\Lambda^2 + p^2)}$$

$$V_0 G V_2 \sim \frac{\Lambda^2}{\Lambda_b^2} \frac{p^2}{(M_\pi^2 + p^2)} \sim O(Q^0)$$

$$\boxed{\int dp', \qquad p' \sim \Lambda}$$

$$\left| V(p', p) - V(p', 0) \right| \cancel{\leq} \left| \frac{p}{p'} \right| \times (\ldots) \text{ if } |p'| > |p|$$

→ Long-range power-counting-breaking terms

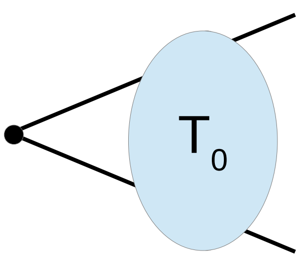
→ Nonrenormalizability (in terms of local counter terms)

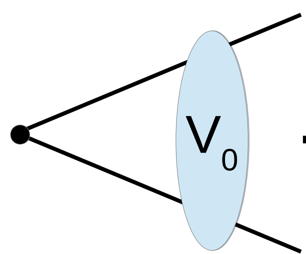
Renormalization in the non-perturbative regime

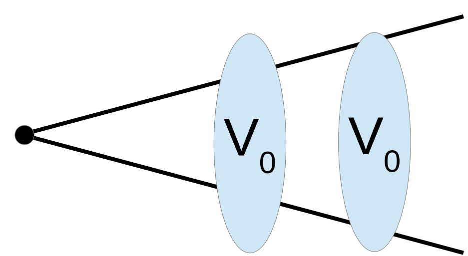
AG, E.Epelbaum, **PRC107**, 044002 (2023)

The series for $\mathbb{R}(T_2^{[m,n]})$ can be summed explicitly:

$$\mathbb{R}\left(T_2\right)(p) = \sum_{m,n=0}^{\infty} \mathbb{R}\left(T_2^{[m,n]}\right)(p) = T_2(p) + \delta C_0\, \psi_p(0)^2, \qquad \delta C_0 = -\frac{T_2(0)}{\psi_0(0)^2}$$

$\psi_p(0)$ =1+


=1+


+


+...

Using Fredholm formula to match to the perturbative regime

$$T_2(p) = (1 + T_0 G)V_2(1 + GT_0) = \frac{N_2(p)}{D(p)^2}$$

$D(p)$-Fredholm determinant

Convergent series in V_0: $N_2 = \sum_{i=0}^{\infty} N_2^{[i]}\,,\; D = \sum_{i=0}^{\infty} D^{[i]}$

The same for the counter terms:

$$\delta T_2 = (1 + T_0 G)\delta V_0^{ct}(1 + GT_0) = \delta C_0 \left[\psi_p(0)\right]^2$$

$$\psi_p(0) = \frac{\nu(p)}{D(p)} \qquad \nu(p) = \sum_{i=0}^{\infty} \nu^{[i]}(p)$$

Renormalizability constraints

$$\mathbb{R}\left(T_2\right)(p) = \frac{1}{D(p)^2}\left[N_2(p) + \delta C_0\,\nu(p)^2\right], \qquad \delta C_0 = -\frac{N_2(0)}{\nu_0^2}$$

Renormalizability constraints

$$\mathbb{R}\,(T_2)\,(p) = \frac{1}{D(p)^2}\left[N_2(p) + \delta C_0\,\nu(p)^2\right], \qquad \delta C_0 = -\frac{N_2(0)}{\nu_0^2}$$

Can be small, or ~0

Renormalizability constraints

$$\mathbb{R}\,(T_2)\,(p) = \frac{1}{D(p)^2}\left[N_2(p) + \delta C_0\,\nu(p)^2\right], \qquad \delta C_0 = -\frac{N_2(0)}{\nu_0^2}$$

Can be small, or ~0

Renormalizability constraints on (the short-range part of) the LO potential.
The simplest formulation: LECs must be of natural size (If $\Lambda \sim \Lambda_b$).

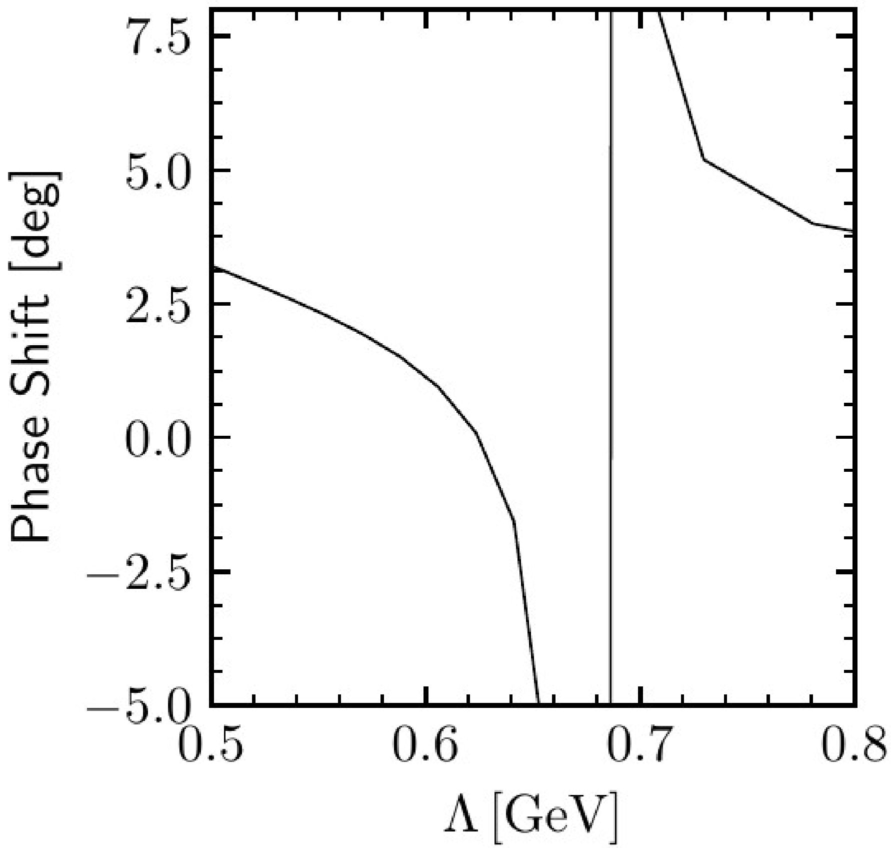
Constraints on the choice of the cutoff are not driven by data!

For realistic interactions this requirement is fulfilled for 450 MeV<Λ<750 MeV
$\Delta\Lambda \sim 2M_\pi \sim Q$, not unnaturally small

Failure of renormalizability for $\Lambda > \Lambda_b, {}^3P_0$

AG, E.Epelbaum, **PRC107**, 034001 (2023)

T_{lab}=130MeV



Sharp cutoff, harder than smooth regulators
0.7 GeV → above 1 GeV

(In)Consistency of Weinberg power counting and its modifications

Large cutoff arguments do not work for the finite cutoff scheme:
Divergencies → positive (uncompensated) power powers of Λ

Mismatch of ultraviolet divergencies and infrared power counting is typicall:
covariant ChPT in the 1-nucleon sector, especially Δ-full
→scheme dependence as a higher order effect

Consistent in the EFT sense: systematic expansion preserving symmetries

---

$^1S_0$,$^3P_0$ partial waves: formally higher order contributions appear large
→ promote to LO to make the scheme more efficient
→ more reliable error estimate

Summary

Explicit renormalization of an EFT provides a justified systematic expansion of observables and theoretical error estimate

Sufficient conditions for renormalizability:

(1) Locality of the long-range forces

(2) Cutoff of the order of the hard scale $\Lambda \approx \Lambda_b$

(3) Naturalness of the counter terms

Outlook

Extend the analysis to other channels (e.g. currents) and higher orders