

# Long-range forces in a finite volume

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- Introduction: where are long-range forces important?
- How does one deal with long-range forces on the lattice?
- EFT and the modified effective range expansion
- Modified Lüscher equation
- Analysis of data in different approaches
- Conclusion, outlook

# Why long-range forces?



- Left hand cut close to threshold: the energy levels below the left-hand branch point cannot be used
- Slowly converging partial-wave expansion: expecting strong admixture of higher partial waves in the quantization condition (Meng & Epelbaum, 2021)
- Exponentially suppressed corrections still sizable

### Left-hand cut: case of NN scattering



• Phase shift real below the left-hand branch point?

### Plane-wave basis (Meng & Epelbaum, 2021)

• Describe the system in terms of the parameters of the effective Lagrangian which, by definition, encode only faraway singularities

$$V(\boldsymbol{p},\boldsymbol{q}) = -\left(\frac{g_A}{2F_{\pi}}\right)^2 \frac{(\boldsymbol{\sigma}_1 \cdot \boldsymbol{k})(\boldsymbol{\sigma}_2 \cdot \boldsymbol{k})}{M_{\pi}^2 + \boldsymbol{k}^2} + C_S + \frac{C_1}{4} (\boldsymbol{p} + \boldsymbol{q})^2 + C_2 \boldsymbol{k}^2 + \cdots$$
$$\boldsymbol{k} = \boldsymbol{p} - \boldsymbol{q}$$

- Work in the plane wave basis; do not resort to the partial-wave expansion
  - For the *NN* scattering, it was shown that, at the physical quark masses, the partial-wave mixing is sizable (Meng & Epelbaum, 2021)
  - A consistent fit of the *DD*<sup>\*</sup> scattering phases to lattice data in the left-hand cut region has been performed (Meng *et al.*, 2023)

# Splitting long- and short-range interactions (Hansen & Raposo, 2023)

- Short-range interactions are re-summed to all orders in the presence of the long-range potential, restricted on-shell
- Fit short-range part to the scattering data, get full amplitude through solving integral equations
  - Involves the *on-shell* short-range amplitude
- Quantization condition is written down both in the plane-wave basis and the partial-wave basis

### Applying three-particle formalism (Hansen et al., 2024)

The limit of a stable  $D^*$  in the  $DD^*$  molecule (plane wave basis is used):

$$\mathscr{M} = (\mathbf{G} + \mathbf{H}) + \frac{1}{L^3} \sum (\mathbf{G} + \mathbf{H}) \tau \mathscr{M}$$

 $G \rightarrow$  one pion exchange,  $H \rightarrow$  short-range force

$$\tau = (p^* \cot \delta(p^*) - J_L(p^*))^{-1} = Z(p^*) \left( {p^{*2} - p_d^{*2}} \right)^{-1}$$

 $s^*~=~(P-q_D)^2\leq (E-m_D)^2~~$  in the CM frame

• Two-particle QC for a stable  $D^*$ , in the vicinity of the  $DD^*$  bound state

•  $D\pi$  channel opens at larger values of  $E \geq 2m_D + M_\pi$ , still close to  $m_{D^*} + m_D$ 

Differences between two- and three-particle frameworks may indicate the importance of the internal structure of  $D^*$  in the analysis of data

## Alternative approaches

- Using Lüscher equation plus EFT with long-range force in the infinite volume above the left-hand cut (Collins *et al.*, 2024)
  - Exponentially suppressed corrections might still be essential just above the left-hand cut.
- HAL QCD approach (Lyu et al., 2023)

### Modified effective range expansion (van Haeringen & Kok, 1982)

- Lüscher equation is based on the assumption  $R \sim M^{-1} \ll L$ ...violated by a long-range force with a small M!
- Splitting of the potential

$$V(r) = \underbrace{V_L(r)}_{\text{known, local}} + \underbrace{V_S(r)}_{\text{unknown}}$$

• Effective-range expansion: very small radius of convergence

$$q^{2\ell+1}\cot\delta_\ell(q) = -rac{1}{a_\ell} + rac{1}{2}\,r_\ell q^2 + O(q^4)$$

• Define modified effective-range function:

$${\cal K}_\ell^{\cal M}(q^2) = {\cal M}_\ell(q) + rac{q^{2\ell+1}}{|f_\ell(q)|^2} \; (\cot(\delta_\ell(q) - \sigma_\ell(q)) - i)$$

#### Jost functions and all that

• Jost function for the long-range interaction:

$$f_{\ell}(q) = rac{q^{\ell} e^{-i\ell\pi/2}(2\ell+1)}{(2\ell+1)!!} \lim_{r o 0} r^{\ell} f_{\ell}(q,r)$$

• The function  $M_{\ell}(q)$ :

$$M_\ell(q) = rac{1}{\ell!} \, \left( -rac{iq}{2} 
ight)^\ell \lim_{r o 0} rac{d^{2\ell+1}}{dr^{2\ell+1}} \, rac{f_\ell(q,r)}{f_\ell(q)}$$

• Larger radius of convergence for the modified effective-range function:

$$K^{M}_{\ell}(q^2) = -rac{1}{\widetilde{a}_{\ell}} + rac{1}{2}\,\widetilde{r}_{\ell}q^2 + O(q^4)$$

• Relation between  $K^M_\ell(q^2)$  and the full phase  $\delta_\ell(q)$  is algebraic

- The long-range potential  $V_L(r)$  is local
- The long-range potential must be superregular

 $\left|\lim_{r\to 0}r^{-2\ell}V_L(r)\right|<\infty$ 

• The short-range potential is a low-energy polynomial:

$$\langle \mathbf{p} | V_S | \mathbf{q} \rangle = C_0^{00} + 3C_1^{00}\mathbf{p}\mathbf{q} + C_0^{10}(\mathbf{p}^2 + \mathbf{q}^2) + \cdots$$

#### The Yukawa potential

• Pauli-Villars regularization:  $V_L(r) = \frac{ge^{-M_{\pi}r}}{r} - \sum_{i=1}^{2\ell+1} c_i \frac{ge^{-M_ir}}{r}$ 

$$M_i \sim M$$
 (heavy scale),  $M_\pi^k = \sum_{i=1}^{2\ell+1} c_i M_i^k$  for  $k = 0, \cdots, 2\ell$ 

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• Sharp cutoff: 
$$V_L(r) = \theta(r - r_0) \frac{g e^{-i r_0 r}}{r}$$

• Warning for the smooth-cutoff-lovers: for example,

$$V_L(r) = exp\left(-\frac{1}{\Lambda r}\right) \frac{ge^{-M_{\pi}r}}{r}$$

... is not acceptable:  $V_L^{\text{reg}}(r) - V_L(r)$  is not a low-energy polynomial with the coefficients of natural size!

$$T = T_L + (1 + T_L G_0) T_S (1 + G_0 T_L)$$
  
$$T_S = V_S + V_S G_L T_S$$

• The Green function with the long-range potential only:  $G_L = G_0 + G_0 V_L G_L$ 



### The loop with infinite number of long-range insertions



$$\langle \boldsymbol{r} | G_L(q_0^2) | \boldsymbol{r}' \rangle = 4\pi \sum_{\ell m} \mathscr{Y}_{\ell m}(\boldsymbol{r}) \tilde{G}_L^\ell(r, r'; q_0^2) \mathscr{Y}_{\ell m}^*(\boldsymbol{r}'), \qquad \langle G_L^\ell(q_0^2) \rangle = \lim_{r, r' \to 0} G_L^\ell(r, r'; q_0^2)$$

• Relation to the Jost functions:

$$\langle G_{L}^{\ell}(q_{0}^{2}) \rangle = \frac{1}{4\pi((2\ell+1)!!)^{2}} M_{\ell}(q_{0}) + \underbrace{\text{real low-energy polynomial in } q_{0}^{2}}_{\text{renormalization prescription}}$$

### Modified effective range expansion: EFT framework

• Lowest order:  $\langle \boldsymbol{p} | V_S | \boldsymbol{q} \rangle = C_0^{00}$ 

$$\underbrace{\frac{4\pi/C_0^{00}}{|f_0(q_0)|^2}}_{=K_0^M(q_0^2) \text{ at lowest order}} = M_0(q_0) + \frac{q_0}{|f_0(q_0)|^2} \left(\cot(\delta_0(q_0) - \sigma_0(q_0)) - i\right)$$

- Higher orders:
  - The quantity K<sub>0</sub><sup>M</sup>(q<sub>0</sub><sup>2</sup>) is a low-energy polynomial in q<sub>0</sub><sup>2</sup>, expressed in terms of couplings C<sub>0</sub><sup>00</sup>, C<sub>1</sub><sup>00</sup>, C<sub>1</sub><sup>00</sup>, ...
  - In the proof, the locality of  $V_L(r)$  plays crucial role. The proof is not valid for a general, non-local potential

#### Modified Lüscher equation

$$\det \mathscr{A}_{\ell m,\ell'm'} = 0, \qquad \mathscr{A}_{\ell m,\ell'm'} = \delta_{\ell\ell'}\delta_{mm'}K^{M}_{\ell}(q_0^2) - H_{\ell m,\ell'm'}(q_0)$$

• Modified Lüscher zeta-function, finite volume:



• Taking into account the renormalization prescription:

$$H_{\ell m,\ell' m'}(q_0) = (H_{\ell m,\ell' m'}(q_0) - H^{\infty}_{\ell m,\ell' m'}(q_0)) + rac{1}{4\pi} \, \delta_{\ell\ell'} \delta_{mm'} M_\ell(q_0)$$

### Shift of the energy levels due to $V_L$ only



## Modified Lüscher function (S-waves only)



### Perturbative shift of the ground-state energy (S-wave only)



### Analysis of data

- Partial-wave mixing in a finite volume is expected to be significantly reduced
- For a single partial wave:

#### Energy level $\rightarrow$ scattering phase at a given energy

• A parameterization of phase shifts in a restricted energy interval is needed, if partial-wave mixing is included.

Solution in the plane wave basis:

- Conceptually, very straightforward and transparent
- Parameterization of the infinite-volume amplitude in terms of the effective couplings is assumed in the whole energy range
- Does this constitute to a disadvantage? Not necessarily...

### Conclusions, outlook

- A novel quantization condition in the presence of the long-range forces has been proposed
  - Solves the left-hand cut problem
  - Reduces partial-wave mixing
  - Relates the energy level to the scattering phase(s) at the same energy
- $\bullet$  Three-body  $\rightarrow$  two-body description for stable dimers
- Long-range force in the three-body quantization condition: e.g., NN in NNN
- Long-range three-body force?
- Electromagnetic interactions: is the non-perturbative resummation of the Coulomb photon exchanges needed/possible?