Roy-Steiner-equation analysis of pion-nucleon scattering and nucleon resonances

J. Ruiz de Elvira

Complutense University of Madrid and IPARCOS

In collaboration with:

M. Hoferichter, B. Kubis, U.-G. Meißner.

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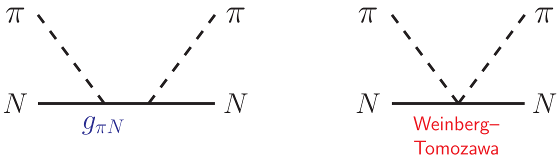
Outline:

Dispersive analysis based on Roy-Steiner equations

- Roy-Steiner solutions: phase-shift — Phys. Rept. 625 (2016) 1-88
- σ-term and comparison to lattice results — Phys. Rev. Lett. 115 (2015) 092301, Phys. Lett. B 760 (2016) 74-78, J. Phys. G 45 (2018) 2, 024001, Phys. Lett. B 843 (2023) 138001
- ChPT matching and chiral low-energy constants — Phys. Rev. Lett. 115 (2015) 19, 192301, Phys. Lett. B 770 (2017) 27-34 (with D. Siemens, E. Epelbaum, and H. Krebs)
- **nucleon resonances** — Phys. Lett. B 853 (2024) 138698

Motivation: chiral πN interactions

- simplest process for chiral pion interaction with nucleons



- leading order $\mathcal{O}(p) = \mathcal{O}(M_\pi)$ free-parameter chiral prediction
  - ▷ scattering lengths

$$a^- = \frac{M_\pi m_N}{8\pi(M_\pi + m_N)F_\pi^2} + \mathcal{O}(M_\pi^3), \qquad a^+ = \mathcal{O}(M_\pi^2)$$

[Weinberg (1966)]

  - ▷ Goldberger-Treiman relation

$$g_{\pi N} = \frac{g_A m_N}{F_\pi}$$

[Goldberger-Treiman (1958)]

Motivation: chiral πN interactions

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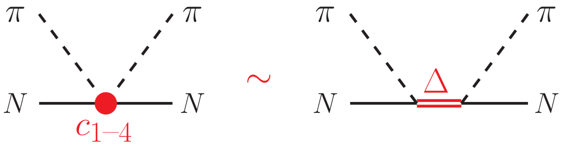
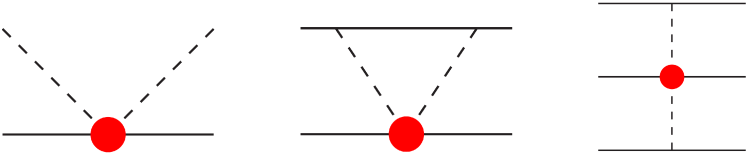


- next-to-leading order $\mathcal{O}(p^2)$: low-energy constants (LECs) $c_1 - c_4$

  effectively incorporate the effect of the $\Delta(1232)$: $m_\Delta - m_N \simeq 2M_\pi$

- c_i very important for nuclear physics: long-range part of NN and $3N$ potential



Motivation: the pion-nucleon σ-term

- **scalar form factor** of the nucleon

$$\sigma(t) = \langle N(p')|\hat{m}\left(\bar{u}u + \bar{d}d\right)|N(p)\rangle \qquad t = (p' - p)^2$$

$$\sigma_{\pi N} \equiv \sigma(0) = \frac{\hat{m}}{2m_N}\langle N|\bar{u}u + \bar{d}d|N\rangle$$

- $\sigma_{\pi N}$ determines the **light-quark contribution** to the nucleon mass

  ▷ Feynman-Hellmann theorem

$$\sigma_{\pi N} = \hat{m}\frac{\partial m_N}{\partial \hat{m}}$$

- relevant for **direct-detection dark matter** searches

  ▷ $\sigma_{\pi N}$ determines the scalar coupling of the nucleon

$$\sigma_{SI} = \frac{4\mu_N^2}{\pi}\left|m_N \sum_q C_q^{SS} f_q^N + \cdots\right|, \qquad \mu_N = \frac{m_N m_\chi}{(m_N + m_\chi)}$$

$$\sigma_{\pi N} = m_N\left(f_u + f_d\right)$$

The σ-term and πN scattering

- no scalar probe, but still relation to experiment

  $\hookrightarrow$ low-energy theorem

  [Cheng, Dashen (1971); Brown, Pardee, Peccei (1971)]

- $\sigma_{\pi N}$ related to πN scattering amplitude

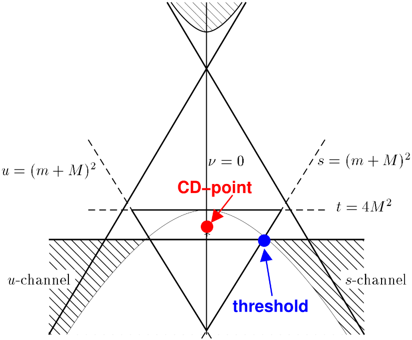
  but at **unphysical kinematics**

$$\underbrace{F_\pi^2 \bar{D}^+(\nu = 0, t = 2M_\pi^2)}_{F_\pi^2 (d_{00}^+ + 2M_\pi^2 d_{01}^+) + \Delta_D} = \underbrace{\sigma(2M_\pi^2)}_{\sigma_{\pi N} + \Delta_\sigma} + \Delta_R$$

- $|\Delta_R| \lesssim 2$ MeV small [Bernard, Kaiser, Meißner (1996)]

  no chiral logs at one-loop order

- $\Delta_D - \Delta_\sigma = (-1.8 \pm 0.2)$ MeV [Hoferichter et al. (2012)]



- need to determine subthreshold parameters d_{00}^+, d_{01}^+

  $\hookrightarrow$ **Roy-Steiner equations**

Roy–Steiner equations for πN: flow of information

Limited range of validity

$$\sqrt{s} \leq \sqrt{s_m} = 1.38\,\mathrm{GeV} \qquad \sqrt{t} \leq \sqrt{t_m} = 2.00\,\mathrm{GeV}$$

Input/Constraints

- S- and P-waves above matching point $s > s_m$ $(t > t_m)$
- inelasticities
- higher waves (D-, F-, $\cdots$)
- scattering lengths from hadronic atoms

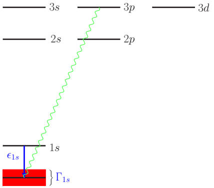
[Baru et al. 2011]

Output

- S- and P-wave phase-shifts at low energies $s < s_m$ $(t < t_m)$
- subthreshold parameters
  - ▷ pion-nucleon σ-term
  - ▷ ChPT LECs
- **nucleon resonances**

Hadronic atoms: constraints for πN

- $\pi H/\pi D$: bound state of π^- and p/d spectrum sensitive to **threshold** πN amplitude [PSI (1995-2010)]



- combined analysis of πH and πD:

$$a_0^+ \equiv a^+ = (7.5 \pm 3.1) \cdot 10^{-3} M_\pi^{-1}$$
$$a_0^- \equiv a^- = (86.0 \pm 0.9) \cdot 10^{-3} M_\pi^{-1}$$

- but: a^+ very sensitive to isospin breaking, PWA based on $\pi^\pm p$ channels

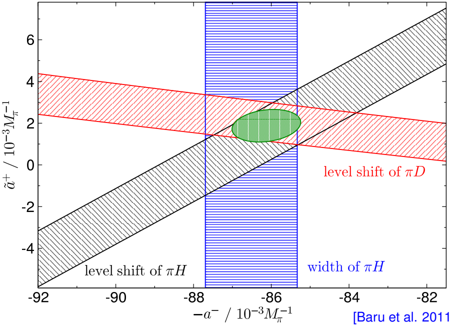
  $\hookrightarrow$ use instead

$$\frac{a_{\pi^- p} + a_{\pi^+ p}}{2} = (-0.9 \pm 1.4) \cdot 10^{-3} M_\pi^{-1}$$

▷ isospin breaking in $\sigma_{\pi N}$ could be important

$$\bar{a}_{0+}^{1/2} = 169.8(2.0) \times 10^{-3} M_\pi^{-1}$$

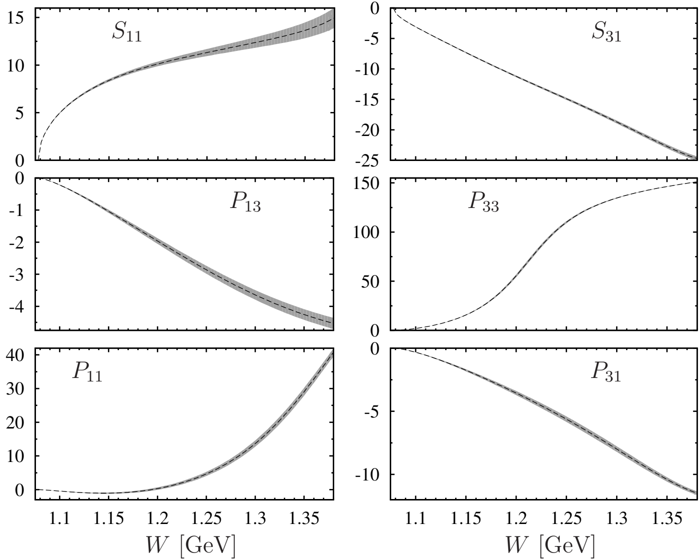
$$\bar{a}_{0+}^{3/2} = -86.3(1.8) \times 10^{-3} M_\pi^{-1}$$



[Baru et al. 2011]

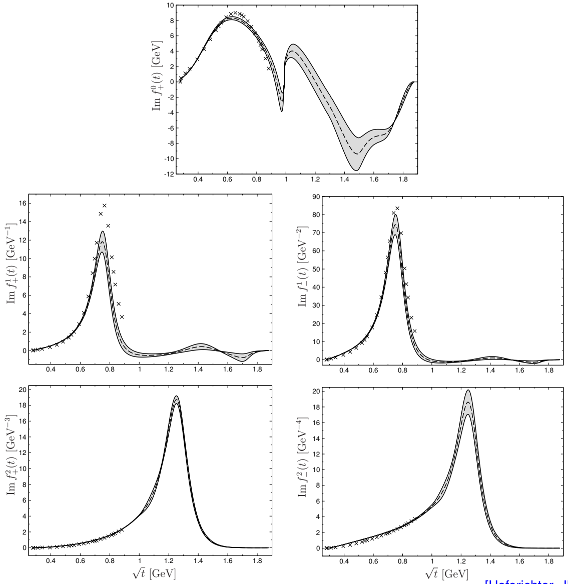
$$\tilde{a}^+ = a^+ + \frac{1}{1 + M_\pi / m_p} \left\{ \frac{M_\pi^2 - m_{\pi 0}^2}{\pi F_\pi^2} c_1 - 2\alpha f_1 \right\}$$

Roy-Steiner solutions: s-channel phase shifts



[Hoferichter, JRE, Kubis, Meißner (2016)]

Roy-Steiner solutions: imaginary part t-channel partial waves



[Hoferichter, JRE, Kubis, Meißner (2016)]

Roy-Steiner solutions: the sigma-term

$$\sigma_{\pi N} = F_\pi^2 \left(d_{00}^+ + 2M_\pi^2 d_{01}^+ \right) + \Delta_D - \Delta_\sigma - \Delta_R$$

- subthreshold parameters output of the Roy–Steiner equations

$$d_{00}^+ = -1.36(3) M_\pi^{-1}, \quad d_{01}^+ = 1.16(2) M_\pi^{-3}$$

- $\Delta_D - \Delta_\sigma = -(1.8 \pm 0.2)$ MeV [Hoferichter at al. (2012)]

  $|\Delta_R| \lesssim 2$ MeV [Bernard, Kaiser, Meißner (1996)]

- isospin breaking in the CD theorem shifts $\sigma_{\pi N}$ by $+3.0$ MeV
- final results:

$$\sigma_{\pi N} = (59.0 \pm 1.6_{\mathrm{SL}} \pm 0.9_{\mathrm{RS}} \pm 2.0_{\mathrm{LET}} \pm 2.2_{\mathrm{IB}})\ \text{MeV} = (59.0 \pm 3.5)\ \text{MeV}$$

[Hoferichter, JRE, Kubis, Meißner (2015)]

Collaboration	χQCD	JLQCD	BMWc	ETMC	RQCD	Mainz
Year	2016	2018	2020	2020	2023	2023
$\sigma_{\pi \mathbf{N}}$ (MeV)	45.9(7.4)(2.8)	26(3)(5)(2)	42.4(3.4)(4.7)	41.6(3.8)	43.9(4.7)	43.7(3.6)
Tension	1.5σ	4.7σ	3.8σ	2.5σ	2.6σ	3.1σ

FAQ 1: Could it be isospin-breaking corrections?

- define isospin amplitudes by $\pi^{\pm}p$ channel

  $\hookrightarrow$ this is what (mainly) enters the PWAs

- calculate isospin-breaking corrections in ChPT, main effect $\Delta_\pi = M_{\pi^\pm}^2 - M_{\pi^0}^2$

- $\bar{a}_{0+}^{1/2}$ and $\bar{a}_{0+}^{3/2}$ from pionic atoms defined consistent with these conventions

  $\hookrightarrow$ large isospin-breaking effect in a_{0+}^{+} [Gasser et al. (2002)]

- for the σ term

$$\sigma_{\pi N} = F_\pi^2\left(d_{00} + 2M_\pi^2 d_{01}\right) - \underbrace{\Delta_R}_{\lesssim 2\text{MeV}} + \underbrace{\Delta_D - \Delta_\sigma}_{(-1.8\pm0.2)\text{MeV}} + \underbrace{\frac{81 g_a^2 M_\pi \Delta_\pi}{256\pi F_\pi^2}}_{+3.4\text{ MeV}} + \underbrace{\frac{e^2}{2} F_\pi^2 (4f_1 + f_2)}_{(-0.4\pm2.2)\text{ MeV}}$$

$$= F_\pi^2\left(d_{00} + 2M_\pi^2 d_{01}\right) + 1.2(3.0)\text{ MeV}$$

  $\hookrightarrow$ main corrections from Δ_π increases $\sigma_{\pi N}$

- lattice convention for the σ-term defined at $M_{\pi^0} \equiv \bar{\sigma}_{\pi N}$

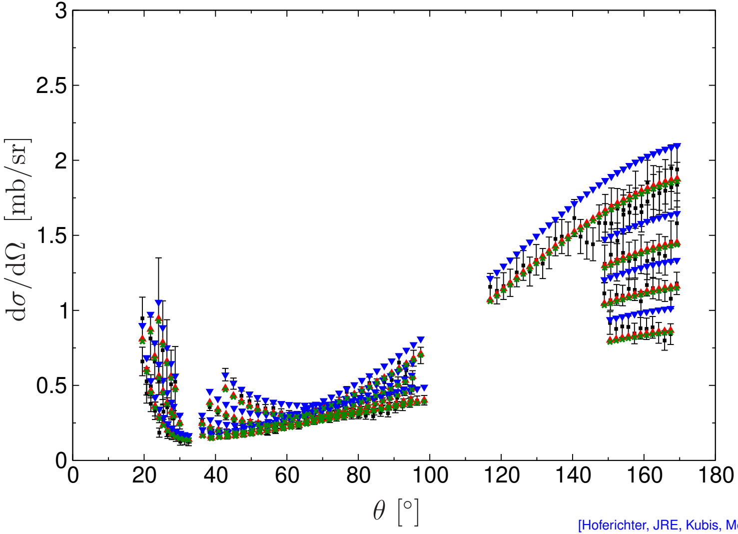
  $\hookrightarrow$ $\Delta\sigma = \sigma_{\pi N} - \bar{\sigma}_{\pi N} = 3.1(5)$ MeV effect! [Hoferichter, JRE, Kubis, Meißner (2023)]

FAQ 2: What if the pionic-atom measurements are wrong?

- fair enough, that's why we looked at cross sections
- challenges in extracting $\sigma_{\pi N}$ from low-energy cross sections
  - ▷ normalizations
  - ▷ electromagnetic corrections
- cannot use existing compilations due to bias from respective fit model
- strategy:
  - ▷ Roy-Steiner representation with scattering length as free parameter
    - $\hookrightarrow$ separately for $\pi^+ p \to \pi^+ p$, $\pi^- p \to \pi^- p$, $\pi^- p \to \pi^0 n$
  - ▷ normalizations as additional fit parameters (with GW as starting point)
  - ▷ keep Coulomb piece of Tromborg correction, consider the rest as error estimate

[Tromborg et al. (1977)]

FAQ 2: What if the pionic-atom measurements are wrong?

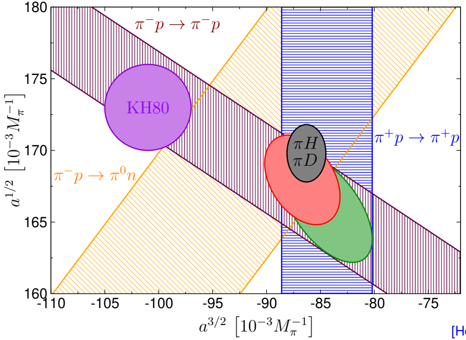


[Hoferichter, JRE, Kubis, Meißner (2017)]

- example from $\pi^+ p \to \pi^+ p$: KH80, pionic atoms, fit to data

  $\hookrightarrow$ can see by eye that **KH80 is disfavored**

FAQ 2: What if the pionic-atom measurements are wrong?



[Hoferichter, JRE, Kubis, Meißner (2017)]

	$a_{0+}^{1/2}\ \left(10^{-3}M_\pi^{-1}\right)$	$a_{0+}^{3/2}\ \left(10^{-3}M_\pi^{-1}\right)$	$\sigma_{\pi N}$ (MeV)
KH80	173(3)	-101(4)	47(5)
pionic atoms	169.8(2.0)	-86.3(1.8)	**59.0(3.5)**
all channels	167.9(3.2)	-86.7(3.5)	**58.3(4.2)**
$\pi^\pm p \to \pi^\pm p$	166.0(3.8)	-84.4(4.2)	59.8(4.5)

The πN σ-term: excited state contamination

- two methods to extract $\bar{\sigma}_{\pi N}$ on the lattice
  - ▷ from the two-point function C^{2pt} via the Feynman-Hellmann theorem
  - ▷ from the scalar nucleon form factor three-point function C^{3pt}
- the nucleon couples to the ground state and all its excitations

$$C^{2pt}(\tau, \mathbf{k}) = \sum_i |A_i(\mathbf{k})|^2 e^{-M_i \tau}, \quad C_S^{3pt}(\tau, t) = \sum_{i,j} A_i A_j^* \langle i|S|j\rangle e^{-M_i t} e^{-M_j (\tau - t)}$$

- how to remove excited state contamination?
- LANL 2021 analysis: two fit strategies [Gupta, Park, Hoferichter, Mereghetti, Yoon, Bhattacharya (2021)]
  - ▷ standard strategy: combined fit to C^{2pt} and C^{3pt} with flat priors on M_i
  - ▷ $N\pi$ fit: includes the $N\pi$ or $N\pi\pi$ multihadron state using narrow-width prior for M_1
  - ↪ lattice data not precise enough to decide but

    $\bar{\sigma}_{\pi N}$ changes from $\sim$ 40 MeV to $\sim$ 60 MeV on including the $N\pi$ and $N\pi\pi$ excited states
- ChPT in the euclidian also suggests large contribution of $N\pi$ or $N\pi\pi$ states to $\sigma_{\pi N}$
  - ▷ NLO large due to non-analytic loop effect (large pion coupling to scalar source)
  - ▷ N^2LO large due to the $\Delta(1232)$
  - ↪ both effects go in the same direction!

The πN σ-term: excited state contamination

Variation	$\sigma_{\pi N}$ [MeV]
$M_\pi < 220$ MeV	42.04(1.27)
$M_\pi < 285$ MeV	41.89(67)
no cut in M_π	41.67(44)
$M_\pi < 220$ MeV+ $\mathcal{O}(a)$	41.58(6.58)
$M_\pi < 285$ MeV+ $\mathcal{O}(a)$	39.31(3.15)
no cut in M_π+ $\mathcal{O}(a)$	37.55(1.82)
$M_\pi < 220$ MeV+ $\mathcal{O}(e^{-mL})$	42.45(1.33)
$M_\pi < 285$ MeV+ $\mathcal{O}(e^{-mL})$	42.43(79)
no cut in M_π+ $\mathcal{O}(e^{-mL})$	42.87(59)
$M_\pi < 220$ MeV+ $\mathcal{O}(a)$+ $\mathcal{O}(e^{-mL})$	42.69(6.68)
$M_\pi < 285$ MeV+ $\mathcal{O}(a)$+ $\mathcal{O}(e^{-mL})$	39.38(3.35)
no cut in M_π+ $\mathcal{O}(a)$+ $\mathcal{O}(e^{-mL})$	39.34(2.08)
$M_\pi < 220$ MeV	46.81(1.14)
$M_\pi < 285$ MeV	43.71(62)
no cut in M_π	41.04(39)
$M_\pi < 220$ MeV+ $\mathcal{O}(a)$	51.38(5.87)
$M_\pi < 285$ MeV+ $\mathcal{O}(a)$	45.77(2.73)
no cut in M_π+ $\mathcal{O}(a)$	40.38(1.65)
$M_\pi < 220$ MeV+ $\mathcal{O}(e^{-mL})$	47.21(1.20)
$M_\pi < 285$ MeV+ $\mathcal{O}(e^{-mL})$	44.44(76)
no cut in M_π+ $\mathcal{O}(e^{-mL})$	42.79(56)
$M_\pi < 220$ MeV+ $\mathcal{O}(a)$+ $\mathcal{O}(e^{-mL})$	52.26(5.93)
$M_\pi < 285$ MeV+ $\mathcal{O}(a)$+ $\mathcal{O}(e^{-mL})$	47.13(2.90)
no cut in M_π+ $\mathcal{O}(a)$+ $\mathcal{O}(e^{-mL})$	43.83(1.87)

[Agadjanov et al. (2023)]

- Mainz 2023 result:
  also two strategies for excited state contamination
  - ▷ upper: “window”
  - ▷ lower: “two-state” (closest to “$N\pi$ fit” from LANL)
- lattice data currently cannot decide between both scenarios

  ↪ final result: average, $\bar{\sigma}_{\pi N} = 43.7(3.6)$ MeV
- for “two-state” fit: systematic increase of (5-10) MeV
  when restricting results to low pion masses

  most reliable result arguably $\bar{\sigma}_{\pi N} = 52.3(5.9)$

  ↪ compares well with RS results $\bar{\sigma}_{\pi N} = 55.9(3.5)$ MeV
- possible **solution** to the **σ-term puzzle**?

Low-energy constants: matching to Chiral Perturbation Theory

- matching to ChPT at the subthreshold point:

  one-to-one correspondence between subthreshold parameters and LECs

  ▷ chiral expansion expected to work best at subthreshold point

  ▷ preferred choice for *NN* scattering due to proximity of relevant kinematic regions
- express the subthreshold parameters in terms of the LECs to $\mathcal{O}(p^4)$ in Heavy-baryon
- invert the system to solve for LECs

	NLO	N^2LO	N^3LO
c_1 [GeV^{-1}]	-0.74 ± 0.02	-1.07 ± 0.02	-1.11 ± 0.03
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\bar{d}_1 + \bar{d}_2$ [GeV^{-2}]	—	1.04 ± 0.06	7.42 ± 0.08
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\bar{e}_{14}$ [GeV^{-3}]	—	—	0.89 ± 0.04

[Hoferichter, JRE, Kubis, Meißner (2015)]

- N^3LO enhanced by $g_A^2(c_3 - c_4) = -16\ \text{GeV}^{-1}$
- what's going on with chiral convergence?

Low-energy constants: convergence of the chiral series

a_{0+}^{-} [$10^{-3} M_{\pi}^{-3}$]	heavy-baryon-*NN*		heavy-baryon-πN		covariant	
	Δ-less	Δ-ful	Δ-less	Δ-ful	Δ-less	Δ-ful
LO	79.4	79.4	79.4	79.4	79.4	79.4
NLO	79.4	79.4(0)	79.4	79.4(0)	80.1	81.9(1)
N^2LO	92.2	92.7(10)	92.9	90.5(9)	89.9	81.7(1.2)
N^3LO	68.5	96.3(2.0)	58.6	69.1(1.2)	83.8	83.4(1.0)
pionic atoms	85.4(9)	85.4(9)	85.4(9)	85.4(9)	85.4(9)	85.4(9)

[Siemens, JRE, Epelbaum, Hoferichter, Krebs, Kubis, Meißner (2016)]

study of the chiral convergence with and without Δ(1232) and relativistic corrections

- including the Δ does reduce size of LECs and improve convergence
- further improvement in covariant formulation, but **why?**
- uncertainty of LECs totally dominated by scheme and chiral order

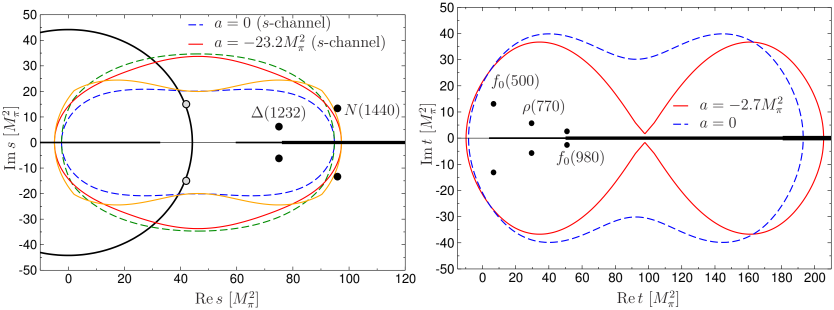
Nucleon resonances

- resonance characterized by
  - ▷ pole on unphysical Riemann sheet (accessible via unitarity cut)
  - ▷ connection to the physical region (not arbitrarily far in the complex plane)
- how to find them?
  - ▷ measurement on the real axis
  - ▷ analytic continuation to an unphysical Riemann sheet
- Non-trivial example: $f_0(500)$
  - ▷ "visible" only as a broad bump in $\pi\pi$ scattering
  - ▷ established from analytic continuation via Roy equations [Caprini, Colangelo, Leutwyler (2006)]
  - ▷ clearly connected to physical region, e.g., via chiral trajectories [Hanhart, Peláez, Ríos (2008)]

$\hookrightarrow$ application of similar ideas to nucleon resonances

Nucleon resonances: Roy-Steiner equations in the complex plane

- Roy-Steiner equations provide model-independent access to the complex plane



- s-channel: $\Delta(1232)$ safely contained, $N(1440)$ borderline, S-wave singularities (gray ovals) close to circular cut

- t-channel: $f_0(500)$, $f_0(980)$, $\rho(770)$ all safely contained

Nucleon resonances: $\Delta(1232)$

- $\Delta(1232)$ is an elastic resonance, located in the second Riemann sheet accessible by crossing continuously the πN cut

$$S^{I}(s+i\epsilon) = S^{II}(s-i\epsilon) \implies S^{II}(s) = 1/S^{I}(s)$$

$\hookrightarrow$ a pole in the second Riemann sheet is a zero in the first one

- $\Delta(1232)$ pole parameters defined as

$$f_{1+}^{3/2}(s)^{II} = \frac{r_\Delta}{|q|(W_\Delta - W)}, \quad W_\Delta = \left(M_\Delta - i\frac{\Gamma_\Delta}{2}\right)$$

- Roy-Steiner result

$$M_\Delta = 1209.5(1.1)\text{ MeV}, \quad \Gamma_\Delta = 98.5(1.2)\text{ MeV}, \quad \rho_{M_\Delta \Gamma_\Delta} = 0.65$$
$$|r_\Delta| = 51.3(9)\text{ MeV}, \quad \delta_{r_\Delta} = -47.4(4)^\circ, \quad \rho_{|r_\Delta|\delta_{r_\Delta}} = 0.11$$

[Hoferichter, JRE, Kubis, Meißner (2024)]

- fully consistent with PDG estimate:

$$M_\Delta = [1209, 1211]\text{ MeV}, \quad \Gamma_\Delta = [98, 102]\text{ MeV}, \quad |r_\Delta| = [49, 52]\text{ MeV}, \quad \delta_{r_\Delta} = [-48, -45]^\circ$$

Nucleon resonances: isospin-breaking corrections for the $\Delta(1232)$

- at this level of precision

  $\hookrightarrow$ isospin breaking becomes relevant!

- RS equations defined by charged channels $\pi^{\pm}p \to \pi^{\pm}p$

  $\hookrightarrow$ results correspond to weighted average of $\Delta^{++} \sim \pi^{+}p$ and $\Delta^{0} \sim \pi^{-}p, \pi^{0}n$

- How we can estimate the $\Delta^{++} - \Delta^{0}$ mass difference?

  $\triangleright$ $m_d - m_u$ corrections computed in heavy-baryon ChPT at NNLO [Tiburzi, Walker-Loud (2006)]

  $\triangleright$ and large-N_c estimates for Δ couplings [Siemens, JRE, Epelbaum, Hoferichter, Krebs, Kubis, Meißner (2016)]

$$m_{\Delta^{++}} - m_{\Delta^0} = \frac{2}{3}(m_p - m_n)\Big|_{\text{QCD}} \times \left\{1 + \frac{2g_A M_\pi^2}{25(4\pi F_\pi)^2}\left[37 + 55\log\left(\frac{M_\pi^2}{\mu^2}\right)\right] + \frac{4g_A^2\Delta^2}{(4\pi F_\pi)^2}\left[\log\left(\frac{4\Delta^2}{M_\pi^2}\right) - \sigma_\Delta \log\left(\frac{1+\sigma_\Delta}{1-\sigma_\Delta}\right)\right]\right\}, \quad \sigma_\Delta = \sqrt{1 - \frac{M_\pi^2}{\Delta^2}}$$

  $\hookrightarrow$ leading to $m_{\Delta^{++}} - m_{\Delta^0}\Big|_{\text{QCD}} = -1.1$ MeV

- radiative corrections?

  $\triangleright$ Cottingham sum rule or lattice?

Nucleon resonances: $N(1440)$

- same procedure for the $N(1440)$ gives

  $M_{N(1440)} = 1473(35)$ MeV PDG: [1360,1380] MeV $\Gamma_{N(1440)} = 73(14)$ MeV PDG: [180,205] MeV

  $\hookrightarrow$ way off from typical Roper parameters, why?

- $N(1440)$ is an inelastic resonance

  $\hookrightarrow$ pole on the third sheet, accessed via $\pi\pi N$ cut, is closer to the physical region!

- RS equations give analytic continuation to the second sheet via πN unitarity cut

  $\hookrightarrow$ we only see the reflection of the true Roper in the second Riemann sheet

Nucleon resonances: $N(1440)$

- possible way around this: use Padé approximants for the analytic continuation
- Benchmark for $\Delta(1232)$:

$$M_\Delta = 1209.5(1.1)\text{ MeV} \to 1209.8(1.5)(0.1)\text{ MeV}, \qquad \Gamma_\Delta = 98.5(1.2)\text{ MeV} \to 98.3(1.7)(0.2)\text{ MeV},$$
$$|r_\Delta| = 51.3(9)\text{ MeV} \to 51.2(2.2)(0.1)\text{ MeV}, \qquad \delta_{r_\Delta} = -47.4(4)^\circ \to 46.8(2.2)(0.2)^\circ,$$

- still competitive within PDG estimate

$$M_\Delta = [1209, 1211]\text{ MeV}, \quad \Gamma_\Delta = [98, 102]\text{ MeV}, \quad |r_\Delta| = [49, 52]\text{ MeV}, \quad \delta_{r_\Delta} = [-48, -45]^\circ$$

- for the $N(1440)$

$$M_{N(1440)} = 1374(3)(4)\text{ MeV}, \qquad \Gamma_{N(1440)} = 215(18)(8)\text{ MeV}$$
$$|r_{N(1440)}| = 58(15)(17)\text{ MeV}, \qquad \delta_{N(1440)} = -65(2)(11)^\circ$$

- Padé uncertainties now substantial, especially for residue [Hoferichter, JRE, Kubis, Meißner (2024)]
- comparing with the PDG

$$M_{N(1440)} = [1360, 1380]\text{ MeV}, \quad \Gamma_{N(1440)} = [180, 205]\text{ MeV}, \quad |r_{N(1440)}| = [50, 60]\text{ MeV}, \quad \delta_{N(1440)} = [-100, -80]^\circ$$

$\hookrightarrow$ suggests a Roper width towards the upper end of the PDG range

Nucleon resonances: subthreshold singularity in the S-wave?

- subthreshold singularities in the S-wave have: a long history
  - ▷ can arise as a consequence of missing LHCs [Döring et al. (2009)]
  - ▷ observed in certain (unitarized) models [Wang et al. (2017), Li, Zheng (2021)]
  - ▷ observed in simplified RS set-up at $M_S = 918(3)$ MeV, $\Gamma_S = 326(18)$ [Cao et al. (2022)]

- we find a pole of $f_{0+}^{1/2}(s)^{II}$ at

$$M_S = 913.9(1.6)\text{ MeV}, \qquad \Gamma_S = 337.7(6.2)\text{ MeV}$$

[Hoferichter, JRE, Kubis, Meißner (2024)]

- how to interpret this?
  - ▷ singularity lies far in the complex plane
  - ▷ singularity essentially sits on the circular cut
  - ▷ connection to the physical region? [ChPT likely not applicable to bridge the gap]

Nucleon resonances: t-channel scalar residues

- t-channel resonance positions input in the RS equations

  but one can obtain residues for their coupling to $\bar{N}N$
- for the scalar resonances $S = f_0(500)$ and $f_0(980)$

$$\frac{g_{SNN}}{g_{S\pi\pi}} = i\sqrt{6}\frac{\sigma_\pi(t_S)}{4m_N^2 - t_S} f_+^0(t_S), \quad \sqrt{t_s} = M_S - \frac{\Gamma_S}{2}, \quad \sigma_\pi(t) = \sqrt{1 - \frac{4M_\pi^2}{t}}$$

  with $f_+^0(t)$ the $\pi\pi \to \bar{N}N$ S-wave on the first sheet
- using consistent $\pi\pi$ Roy-equation input for $f_0(500)$ and $f_0(980)$ pole parameters obtain

$$g_{f_0(500)NN} = 12.1(1.4) - 13.9(5)\, i, \quad g_{f_0(980)NN} = 9.1(9) - 2.9(5)\, i$$

[Hoferichter, JRE, Kubis, Meißner (2024)]

- defining the $f_0(500)$ and $f_0(980)$ decay constants F_S from the pion scalar form factor

  ↪ can test Goldberger-Treiman relations [Carruthers (1971)]

$$\frac{F_{f_0(500)NN}\, g_{f_0(500)NN}}{m_N} = 0.90(28) - 2.78(20)i, \quad \frac{F_{f_0(980)NN}\, g_{f_0(980)NN}}{m_N} = -1.69(27) - 0.25(15)i$$

  ↪ real part for $f_0(500)$ indeed close to 1, but sizable imaginary part

Nucleon resonances: $\rho(770)$ residue

- For the $\rho(770)$

$$\frac{g^{(1)}_{\rho NN}}{g_{\rho\pi\pi}} = -\,2i\sigma_\pi(t_\rho)\frac{m_N(t_\rho - M_\pi^2)}{t_\rho - m_N^2}\left[f^1_+(t_\rho) - \frac{t_\rho}{4\sqrt{2}m_N}f^1_-(t_\rho)\right],$$

$$\frac{g^{(2)}_{\rho NN}}{g_{\rho\pi\pi}} = +\,2i\sigma_\pi(t_\rho)\frac{m_N(t_\rho - M_\pi^2)}{t_\rho - m_N^2}\left[f^1_+(t_\rho) - \frac{m_N}{\sqrt{2}}f^1_-(t_\rho)\right] \qquad \sqrt{t_\rho} = M_{\rho(770)} - \frac{\Gamma_{\rho(770)}}{2}$$

  with $f^1_\pm(t)$ $\pi\pi \to \bar{N}N$ P-waves and $g^{(1)/(2)}_{\rho NN}$ vector/tensor couplings

- using again consistent $\pi\pi$ Roy-equation input for $\rho(770)$ pole parameters obtain

$$g^{(1)}_{\rho NN} = 3.31(69) + 2.99(36)i, \quad g^{(2)}_{\rho NN} = 33.4(2.7) + 9.0(1.0)i$$

[Hoferichter, JRE, Kubis, Meißner (2024)]

- test universality of the $\rho(770)$ couplings

$$\frac{g^{(1)}_{\rho NN}}{g_{\rho\pi\pi}} = 0.50(12) + 0.55(5)i, \quad \frac{g^{(2)}_{\rho NN}}{g_{\rho\pi\pi}} = 1.46(12) + 0.54(3)i$$

  expected to be 1 in the narrow-width limit

  $\hookrightarrow$ sizable universality violation

Summary

- review of Roy–Steiner results for πN
- precise determination of the $\sigma_{\pi N}$
  - ▷ with modern pionic-atom input for scattering lengths $\sigma_{\pi N} = 59.0(3.5)$ MeV
  - ▷ with low-energy πN scattering data $\sigma_{\pi N} = 58(5)$ MeV
- lingering tension with lattice QCD needs to be resolved
  - ▷ related to the analysis excited state contamination?
- extraction of the ChPT LECs
- nucleon resonances
  - ▷ precision determination of $\Delta(1232)$ and $N(1440)$ pole parameters
  - ▷ $\bar{N}N$ residues, $\rho(770)$ universality violation,

    Goldberger-Treiman for $f_0(500)$ and $f_0(980)$ strongly violated

Spare slides

Warm up: Roy-equations for $\pi\pi$ the well-known example

Roy equations = coupled system of partial-wave dispersion relations
+ crossing symmetry + unitarity

- start from twice-subtracted **fixed-t** DRs of the generic form

$$T^I(s,t) = c(t) + \frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'^2}\left[\frac{s^2}{(s'-s)} - \frac{u^2}{(s'-u)}\right]\mathrm{Im}\,T^I(s',t)$$

- subtraction functions c(t) are determined via crossing symmetry

$\hookrightarrow$ functions of the scattering lengths: a_0^0 and a_0^2

- project onto partial waves t_J^I (s) (angular momentum J , isospin I) [Roy (1971)]

expand $\mathrm{Im}\; T^I(s',t)$ in partial waves

$$t_J^I(s) = \text{polynomial}(a_0^0, a_0^2) + \sum_{J'=0}^{\infty}(2J'+1)\sum_{I'=0,1,2}\int_{4m_\pi^2}^{\infty} ds'\, K_{JJ'}^{II'}(s',s)\,\mathrm{Im}\, t_{J'}^{I'}(s')$$

- kernel functions $K_{JJ'}^{II'}(s',s)$ analytically known

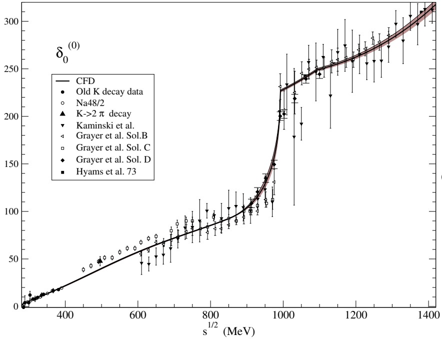
$\pi\pi$ Roy equations: results

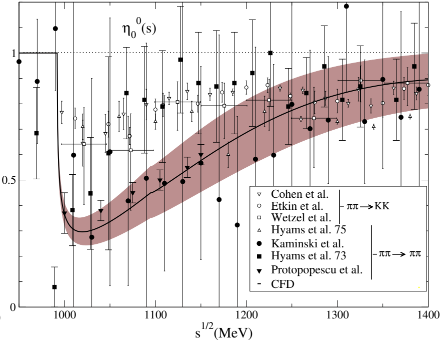
- elastic unitarity

$$t_J^I(s) = \frac{e^{2i\delta_J^I(s)} - 1}{2\,i\,\sigma(s)}, \qquad \sigma(s) = \sqrt{1 - \frac{4M_\pi^2}{s}}$$

$\hookrightarrow$ coupled integral equations for phase shifts

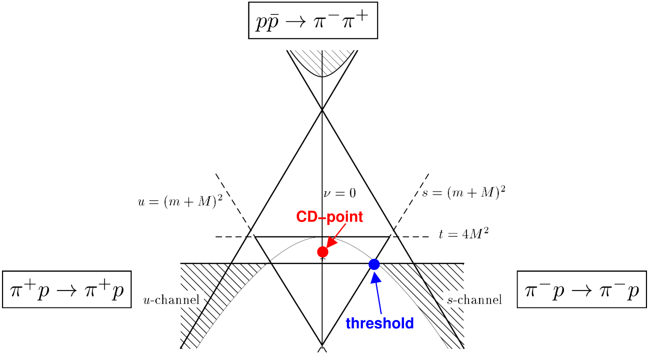
- example: solution for the $\pi\pi$ I=0 S-wave phase shift $\delta_0^0(s)$ and elasticity $\eta_0^0(s)$





[García-Martin, Kaminski, Peláez, JRE, Yndurain (2011)]

Roy–Steiner equations for πN



- **Key challenges:**

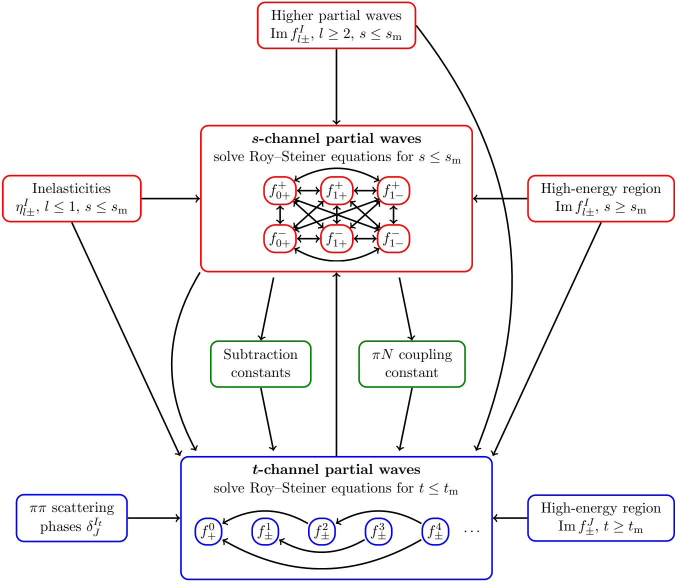
  ▷ crossing: coupling between $\pi N \to \pi N$ and $\pi\pi \to \bar{N}N$

  $\hookrightarrow$ hyperbolic dispersion relations

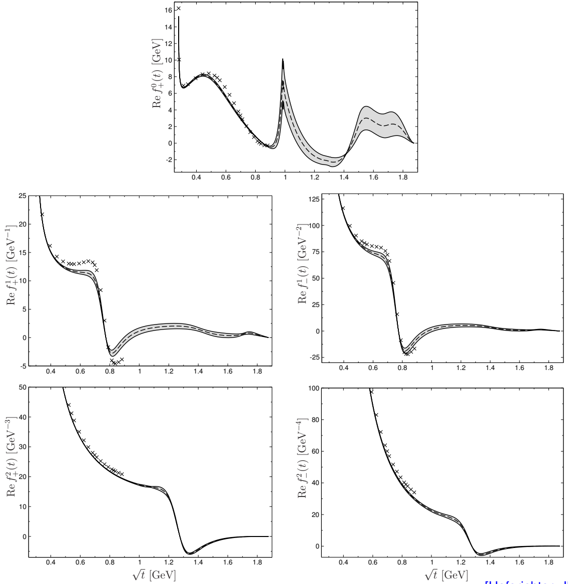
  ▷ unitarity: large pseudophysical region in the t-channel $t = 4M_\pi^2 \longrightarrow 4m_N^2$

  $\hookrightarrow$ $\pi\pi$ and $\bar{K}K$ intermediate states

Roy-Steiner solutions: flow of information



Roy-Steiner solutions: real part t-channel partial waves



[Hoferichter, JRE, Kubis, Meißner (2016)]

Roy-Steiner solutions: uncertainties

- Statistical errors (at intermediate energies)
  - ▷ important correlations between subthreshold parameters
  - ▷ shallow fit minima
  - ↪ Sum rules for subthreshold parameters become essential to reduce the errors

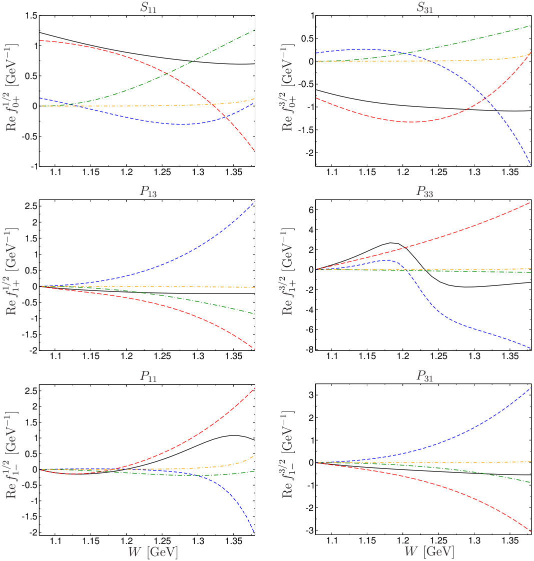
- Input variation (small)
  - ▷ small effect for considering s-channel KH80 input
  - ▷ very small effects from $L > 5$ s-channel PWs
  - ▷ small effect from the different S-wave extrapolation for $t > 1.3$ GeV
  - ▷ negligible effect of ρ' and ρ''
  - ▷ very significant effects of the D-waves ($f_2(1275)$)
  - ▷ F-waves shown to be negligible

- matching conditions (close to W_m)

- scattering length (SL) errors (on S-waves and subthreshold parameters)
  - ↪ very important for the $\sigma_{\pi\mathbf{N}}$

Roy-Steiner solutions: decomposition of the equations

$$f_{l\pm}^{I_s}(W) = N_{l\pm}^{I_s}(W) + K_s T(W)_{l\pm}^{I_s} + K_t T(W)_{l\pm}^{I_s} + DT(W)_{l\pm}^{I_s}$$



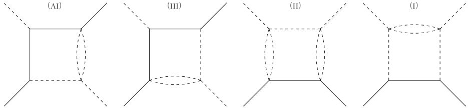
Roy-Steiner solutions: range of convergence

- Assumption: Mandelstam analyticity [Mandelstam (1958,1959)]
  $\Rightarrow$ T(s,t) can be written in terms double spectral densities: ρ_{st}, ρ_{su}, ρ_{ut}

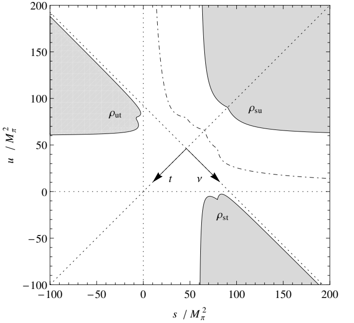
$$T(s,t) = \frac{1}{\pi^2}\iint ds'du' \frac{\rho_{su}(s',u')}{(s'-s)(u'-u)} + \frac{1}{\pi^2}\iint dt'du' \frac{\rho_{tu}(t',u')}{(t'-t)(u'-u)} + \frac{1}{\pi^2}\iint ds'dt' \frac{\rho_{st}(s',t')}{(s'-s)(t'-t)}$$

$\hookrightarrow$ integration ranges defined by the support of the double spectral densities ρ

- Boundaries of ρ are given lowest lying intermediate states



- They limit the range of validity of the HDRS:
- Pw expansion converge
  $\triangleright$ $z = \cos\theta \in$ Lehmann ellipses [Lehmann (1958)]
- the hyperbolae $(s-a)(u-a) = b$
  does not enter any double spectral region
  $\triangleright$ for a value of a, constraints on b yield ranges in s & t



The σ-term: phenomenological status

- **Karlsruhe/Helsinki** partial-wave analysis KH80 [Höhler et al. (1980s)]

  $\hookrightarrow$ comprehensive analyticity constraints, old data

- Formalism for the extraction of $\sigma_{\pi N}$ via the **Cheng–Dashen low-energy theorem**

  $\hookrightarrow$ "canonical value" $\sigma_{\pi N} \sim 45$ MeV, based on KH80 input [Gasser, Leutwyler, Locher, Sainio (1988,1991)]

- **GWU/SAID** partial-wave analysis [Pavan, Strakovsky, Workman, Arndt (2002)]

  $\hookrightarrow$ much larger value $\sigma_{\pi N} = (64 \pm 8)$ MeV

- More recently: ChPT in different regularizations (w/ and w/o Δ) [Alarcón et al. (2012)]

  $\hookrightarrow$ fit to PWAs, $\sigma_{\pi N} = 59 \pm 7$ MeV

- This talk: two new sources of information on low-energy πN scattering

  ▷ Precision extraction of πN scattering lengths from hadronic atoms

  ▷ Roy-equation constraints: analyticity, unitarity, crossing symmetry

FAQ 3: Why does your number differ from Gasser, Leutwyler, Sainio?

- Gasser, Leutwyler, Sainio 1991 relies on Karlsruhe-Helsinki partial-wave analysis "KH80"

  ▷ input comparison

	$g^2/4\pi$	$a_{0+}^{1/2}$ $\left(10^{-3} M_\pi^{-1}\right)$	$a_{0+}^{3/2}$ $\left(10^{-3} M_\pi^{-1}\right)$	$\sigma_{\pi N}$ (MeV)
GLS solution	14.28	173(3)	-101(4)	45
RS solution with HA	13.7(2)	169.8(2.0)	-86.3(1.8)	59.0(3.5)

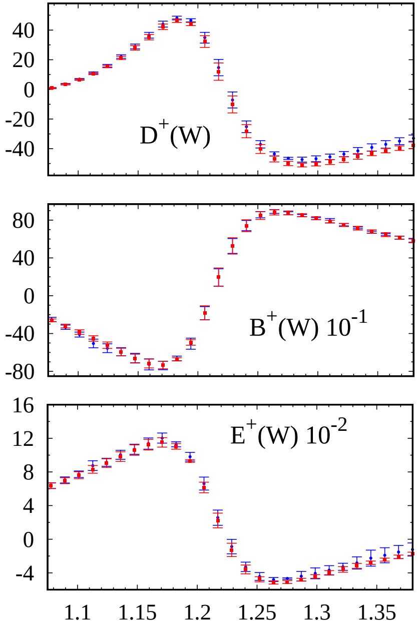
- RS eqs. with KH80 input $\hookrightarrow$ $\sigma_{\pi \mathbf{N}} = 46$ MeV

  $\hookrightarrow$KH80 is internally **consistent** but at odd with the modern SL determinations

- how are d_{00}^+ and d_{01}^+ extracted in Gasser, Leutwyler, Sainio 1991?

  $\hookrightarrow$ Forward Dispersion Relations

FAQ 3: Why does your number differ from Gasser, Leutwyler, Sainio?



blue/red ⇔ LHS/RHS

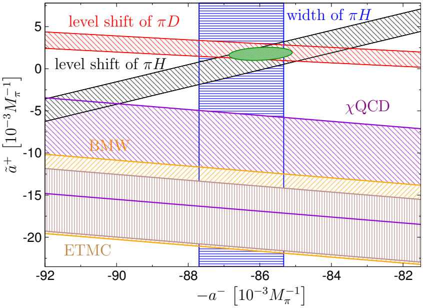
↪ Roy-Steiner equation solutions with pionic atoms satisfy Forward Dispersion Relations

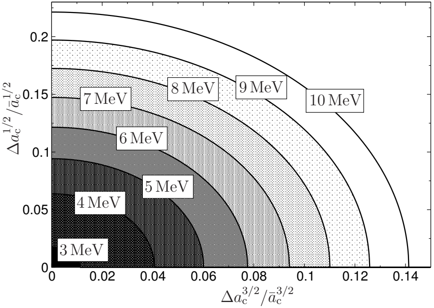
The πN σ-term: comparison to lattice results

- lattice determination of $\sigma_{\pi N}$ at (almost) the physical point

Collaboration	χQCD	JLQCD	BMWc	ETMC	RQCD	Mainz
Year	2016	2018	2020	2020	2023	2023
$\sigma_{\pi \mathbf{N}}$ (MeV)	45.9(7.4)(2.8)	26(3)(5)(2)	42.4(3.4)(4.7)	41.6(3.8)	43.9(4.7)	43.7(3.6)
Tension	1.5σ	4.7σ	3.8σ	2.5σ	2.6σ	3.1σ

- tension can be illustrated in scattering-length plane



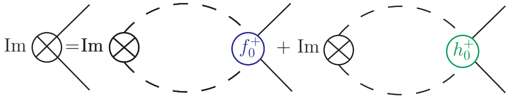


[Hoferichter, JRE, Kubis, Meißner (2016)]

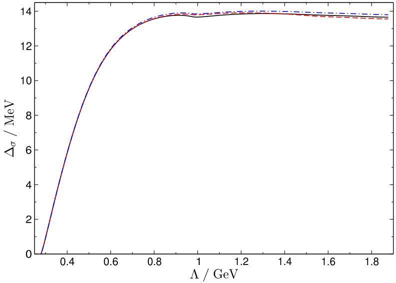
$\hookrightarrow$ independent constraint from lattice calculation of $a_{0+}^{1/2}$ and $a_{0+}^{3/2}$

The σ-term: dispersion relation for the scalar form factor of the nucleon

- Unitarity relation: $\mathrm{Im}\,\sigma(t) = \frac{2}{4m^2-t}\left\{\frac{3}{4}\sigma_t^\pi\left(F_\pi^S(t)\right)^* f_+^0(t) + \sigma_t^K\left(F_K^S(t)\right)^* h_+^0(t)\right\}$



- Once subtracted dispersion relation: $\sigma(t) = \sigma_{\pi N} + \frac{t}{\pi}\int_{t_\pi}^{\infty} \mathrm{d}t' \frac{\mathrm{Im}\sigma(t')}{t'(t'-t)}$



- $\Delta_\sigma = \sigma(2M_\pi^2) - \sigma_{\pi N}$

The σ-term: dispersion relation for the πN amplitude

- t-channel expansion of the subtracted pseudo-Born amplitude

$$\bar{D}(\nu=0,t) = 4\pi \left\{ \frac{1}{p_t^2}\,\bar{f}_0^+(t) + \frac{5}{2} q_t^2\,\bar{f}_2^+(t) + \frac{27}{8} p_t^2 q_t^4\,\bar{f}_4^+(t) + \frac{56}{16} p_t^4 q_t^6\,\bar{f}_6^+(t) + \cdots \right\}$$

- Insert t-channel RS equations for Born-term-subtracted amplitudes $\bar{f}_J^+(t)$

$$\bar{D}(\nu=0,t) = d_{00}^+ + d_{01}^+ t - 16t^2 \int_{t_\pi}^{\infty} \mathrm{d}t' \frac{\mathrm{Im}\bar{f}_0^+(t')}{t'^2(t'-4m^2)(t'-t)} + \{J \geq 2\} + \{\text{s-channel integral}\}$$

- $\Delta_D = F_\pi^2 \left(\bar{D}(\nu=0,t) - d_{00}^+ + d_{01}^+ t\right)$ from evaluation at $t = 2M_\pi^2$

The σ-term: summary of σ-term corrections

- Nucleon scalar form factor

$$\Delta_\sigma = (13.9 \pm 0.3)\,\mathrm{MeV}$$
$$+ Z_1\left(\frac{g^2}{4\pi} - 14.28\right) + Z_2\left(d_{00}^+ M_\pi + 1.46\right) + Z_3\left(d_{01}^+ M_\pi^3 - 1.14\right) + Z_4\left(b_{00}^+ M_\pi^3 + 3.54\right)$$
$$Z_1 = 0.36\,\mathrm{MeV}\,, \qquad Z_2 = 0.57\,\mathrm{MeV}\,, \qquad Z_3 = 12.0\,\mathrm{MeV}\,, \qquad Z_4 = -0.81\,\mathrm{MeV}$$

- πN amplitude

$$\Delta_D = (12.1 \pm 0.3)\,\mathrm{MeV}$$
$$+ \hat{Z}_1\left(\frac{g^2}{4\pi} - 14.28\right) + \hat{Z}_2\left(d_{00}^+ M_\pi + 1.46\right) + \hat{Z}_3\left(d_{01}^+ M_\pi^3 - 1.14\right) + \hat{Z}_4\left(b_{00}^+ M_\pi^3 + 3.54\right)$$
$$\hat{Z}_1 = 0.42\,\mathrm{MeV}\,, \qquad \hat{Z}_2 = 0.67\,\mathrm{MeV}\,, \qquad \hat{Z}_3 = 12.0\,\mathrm{MeV}\,, \qquad \hat{Z}_4 = -0.77\,\mathrm{MeV}$$

$\hookrightarrow$ most of the dependence on the πN parameters cancels in the difference

Full Correction

$$\Delta_D - \Delta_\sigma = (-1.8 \pm 0.2)\ \mathrm{MeV}$$

Goldberger-Miyazawa-Oehme sum rule

- Fixed-t dispersion relations at threshold $\hookrightarrow$ **GMO sum rule**

$$\frac{g^2}{4\pi} = \left(\left(\frac{m_p + m_n}{M_\pi}\right)^2 - 1\right)\left\{\left(1 + \frac{M_\pi}{m_p}\right)\frac{M_\pi}{4}\left(a_{\pi^- p} - a_{\pi^+ p}\right) - \frac{M_\pi^2}{2} J^-\right\}$$

$$= 13.69 \pm 0.12 \pm 0.15$$

$$J^- = \frac{1}{4\pi^2}\int_0^\infty dk \frac{\sigma^{\text{tot}}_{\pi^- p}(k) - \sigma^{\text{tot}}_{\pi^+ p}(k)}{\sqrt{M_\pi^2 + k^2}}$$

- J^- known very accurately [Ericson et al. (2002), Abaev et al. (2007)]

- other determinations

	de Swart et al. 97	Arndt et al. 94	Ericson et al. 02	Bugg et al. 73	KH80
method	NN	πN	GM0	πN	πN
$g^2/4\pi$	13.54 ± 0.05	13.75 ± 0.15	14.11 ± 0.20	14.30 ± 0.18	14.28

- With KH80 scattering lengths $g^2/4\pi = 14.28$ is reproduced exactly

$\hookrightarrow$ discrepancy related to old scattering length values

The σ-term: Cheng-Dashen theorem and isospin breaking

- Define as isoscalar as

$$x^+ \to x^p = \frac{1}{2}\left(X_{\pi^+ p \to \pi^+ p} + X_{\pi^- p \to \pi^- p}\right), \qquad X \in \{D, d_{00}, d_{01}, a_{0+} \ldots\}$$

and "isospin limit" by proton and charged pion

- Assume virtual photons to be removed

$\hookrightarrow$ scenario closest to actual πN PWA

- Calculate IV corrections in SU(2) ChPT, mainly due to $\Delta_\pi = M_\pi^2 - M_{\pi^0}^2$
  - For the σ term no differences at $\mathcal{O}(p^3)$

$$\sigma_{\pi N} = \sigma_p = \sigma_n = -4c_1 M_{\pi^0}^2 - \frac{3g_A^2 M_{\pi^0}^2}{64\pi F_\pi^2}(2M_\pi + M_{\pi^0}) + \mathcal{O}(M_\pi^4)$$

  - Slope of the scalar form factor

$$\Delta_\sigma^p = \sigma_p(2M_\pi^2) - \sigma_p = \frac{3g_A^2 M_\pi^3}{64\pi F_\pi^2} + \frac{g_A^2 M_\pi \Delta_\pi}{128\pi F_\pi^2}\left(-7 + \sqrt{2}\log(3 + 2\sqrt{2})\right) + \mathcal{O}(M_\pi^4)$$

  - Similarly for Δ_D^p

$$\Delta_D^p = F_\pi^2\left\{\bar{D}_p(0, 2M_\pi^2) - d_{00}^p - 2M_\pi^2 d_{01}^p\right\} = \frac{23 g_a^2 M_\pi^3}{384\pi F_\pi^2} + \frac{g_a^2 M_\pi \Delta_\pi}{256\pi F_\pi^2}\left(3 + 4\sqrt{2}\log\left(1 + \sqrt{2}\right)\right) + \mathcal{O}(M_\pi^4)$$

The σ-term: Cheng-Dashen theorem and isospin breaking

- Taking everything together

$$\sigma_{\pi N} = F_\pi^2 \left(d_{00}^p + 2M_\pi^2 d_{01}^p\right) - \Delta_R + \Delta_D - \Delta_\sigma + (\Delta_D^p - \Delta_D) - (\Delta_\sigma^p - \Delta_\sigma)$$
$$+ \sigma_p(2M_\pi^2) + F_\pi^2 \bar{D}(0, 2M_\pi^2)$$
$$= F_\pi^2 \left(d_{00}^p + 2M_\pi^2 d_{01}^p\right) - \underbrace{\Delta_R}_{\lesssim 2\text{MeV}} + \underbrace{\Delta_D - \Delta_\sigma}_{(-1.8\pm 0.2)\text{MeV}} + \underbrace{\frac{81 g_a^2 M_\pi \Delta_\pi}{256\pi F_\pi^2}}_{3.4\text{MeV}} + \underbrace{\frac{e^2}{2} F_\pi^2 (4f_1 + f_2)}_{(-0.4\pm 2.2)\text{MeV}}$$

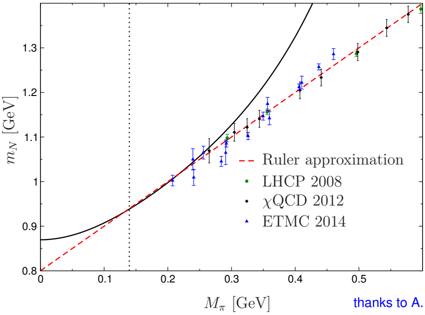
$\hookrightarrow$ sizable corrections from Δ_π increasing the value of the $\sigma_{\pi N}$

The “ruler plot” vs. ChPT

Lattice QCD simulations can be performed at different quark/pion masses

Pion mass dependence of m_N up to NNNLO in ChPT, using

- Input from Roy–Steiner solution



thanks to A. Walker-Loud for providing the lattice data

↪ range of convergence of the chiral expansion is very limited

↪ huge cancellation amongst terms to produce a linear behavior

The σ-term: nucleon strangeness

- relate $\sigma_{\pi N}$ to strangeness content of the nucleon:

$$\sigma_{\pi N} = \frac{\hat{m}}{2m_N} \frac{\langle N|\bar{u}u + \bar{d}d - 2\bar{s}s|N\rangle}{1-y} = \frac{\sigma_0}{1-y}, \quad y \equiv \frac{2\langle N|\bar{s}s|N\rangle}{\langle N|\bar{u}u + \bar{d}d|N\rangle}$$

  $(m_s - m)\left(\bar{u}u + \bar{d}d - 2\bar{s}s\right) \subset$ LQCD produces SU(3) mass splittings:

$$\sigma_{\pi N} = \frac{\sigma_0}{1-y}, \quad \sigma_0 = \frac{\hat{m}}{m_s - \hat{m}} (m_\Xi + m_\Sigma - 2m_N) \sim 26\,\text{MeV}$$

  higher-order corrections: $\sigma_0 \to (36 \pm 7)$ MeV [Borasoy, Meißner (1997)]

- potentially large effects
  - ▷ from the decuplet [Alarcon et al. 2013, Siemens et al. (2016)]
  - ▷ from relativistic corrections (EOMS vs. heavy-baryon)

  $\hookrightarrow$ may increase to $\sigma_0 = (58 \pm 8)$ MeV

- **Conclusion**:
  - ▷ $\sigma_{\pi N} = (59.0 \pm 3.5)$ MeV not incompatible with small y
  - ▷ chiral convergence of σ_0 (hence $\langle N|\bar{s}s|N\rangle$) very doubtful

The σ-term: comparison to experimental cross-section data

Unravel the tension around the σ-term comparing with the experimental πN data base

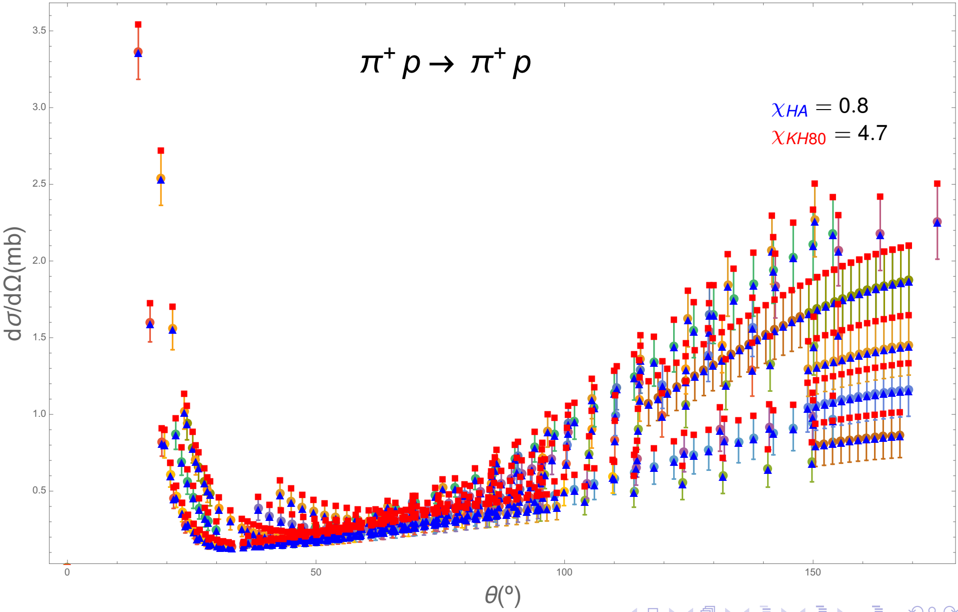
- Generate RS differential cross sections
  - ▷ RS S and P waves
  - ▷ higher partial waves from SAID and KH80 [Workman et al. 2006,2012, Höhler et al. (1980s)]
  - ▷ EM interactions implemented using Tromborg proccedure [Tromborg et al. (1977)]
- Uncertainties from statistical effects, SL, input variation
  - ▷ below $T_\pi = 50$ MeV uncertainties dominated by scattering length errors
  - ↪ disentangle RS SL solutions by looking at the data base
- Define:

$$\chi^2_{a^I_{0+}} = \sum_{i,j} \frac{\left(\mathcal{O}^{\text{exp}}{}_{i,j} - \mathcal{O}^{\text{RS}}_{i,j}(a^I_{0+})\right)^2}{\Delta\mathcal{O}^{\text{exp}\,2}_{i,j}}$$

- Discrepancy concentrated in the $\pi^+ p \to \pi^+ p$ channel

	RS	KH80
$a^{1/2}_{0+}$ $[10^{-3} M_\pi^{-1}]$	169.8 ± 2.0	173 ± 3
$a^{3/2}_{0+}$ $[10^{-3} M_\pi^{-1}]$	-86.3 ± 1.8	-101 ± 4

The σ-term: $\pi^+ p \to \pi^+ p$ cross section



The σ-term: extraction from experimental cross-section data

- Linearized version of RS $d\sigma/d\Omega$ around the HA scattering lengths
- Unbiased fit to the pion-nucleon data base $\Rightarrow$ normalizations constants as fit parameters
- minimize iteratively unbiased χ^2 as a function of a^I_{0+} and ζ

$$\chi^2(a, a_0, \zeta, \zeta_0, \Delta\zeta_0) = \sum_{k=1}^{N} \chi^2_k(a, a_0, \zeta, \zeta_0, \Delta\zeta_0),$$

$$\chi^2_k(a, a_0, \zeta, \zeta_0, \Delta\zeta_0) = \sum_{i,j=1}^{N_k} \left(\zeta_k^{-1}\sigma(W_i^k, a) - \sigma_i^k\right) \left(C_k^{-1}(a_0, \zeta_0, \Delta\zeta_0)\right)_{ij} \left(\zeta_k^{-1}\sigma(W_j^k, a) - \sigma_j^k\right),$$

$$\left(C_k(a_0, \zeta_0, \Delta\zeta_0)\right)_{ij} = \delta_{ij}\left(\Delta\sigma_i^k\right)^2 + \sigma(W_i^k, a_0)\sigma(W_j^k, a_0)\left(\frac{\Delta\zeta_{0,k}}{\zeta^2_{0,k}}\right)^2, \qquad (1)$$

channel	SL combination	result	HA SL	KH80 SL
$\pi^+ p \to \pi^+ p$	$a^{3/2}_{0+}$	-84.4 ± 1.5	-86.3 ± 1.8	-101 ± 4
$\pi^- p \to \pi^- p$	$(2a^{1/2}_{0+} + a^{3/2}_{0+})/3$	82.5 ± 1.5	84.4 ± 1.7	81.6 ± 2.4
$\pi^- p \to \pi^0 n$	$-\sqrt{2}(a^{1/2}_{0+} - a^{3/2}_{0+})/3$	-122.3 ± 3.4	-120.7 ± 1.3	-129.2 ± 2.4

Nucleon form factor: $\pi\pi$ continuum

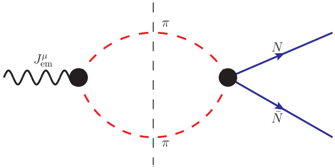
- Electromagnetic nucleon form factor:

$$\langle N(p')|j^\mu_{em}|N(p)\rangle = \bar{u}(p')\left[F_1^N(t)\gamma^\mu + \frac{i\sigma^{\mu\nu}q_\nu}{2m_N}F_2^N(t)\right]u(p)$$

$$G_E^N(t) = F_1^N(t) + \frac{t}{4m_N^2}F_2^N(t), \quad G_M^N(t) = F_1^N(t) + F_2^N(t)$$

- first inelastic correction from $\pi\pi$ continuum

$$\text{Im}\, G_E^v(t) = \frac{q_t^3}{m_N\sqrt{t}}\left(F_\pi^V(t)\right)^* f_+^1(t)\theta(t-t_\pi)$$
$$\text{Im}\, G_M^v(t) = \frac{q_t^3}{\sqrt{2t}}\left(F_\pi^V(t)\right)^* f_-^1(t)\theta(t-t_\pi)$$



- rigorous constraint fixed from:
  - ▷ RS t-channel partial waves
  - ▷ pion form factor

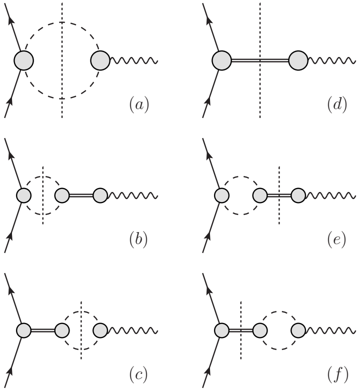
- update of Höhler spectral functions, including also isospin breaking

criticism by Lee et al. (2015)

Nucleon form factor: $\rho - \omega$ mixing

- Isovector and isoscalar nucleon form factor

$$F_i^s(t) = \frac{1}{2}\left(F_i^p(t) + F_i^n(t)\right), \quad F_i^v(t) = \frac{1}{2}\left(F_i^p(t) - F_i^n(t)\right)$$

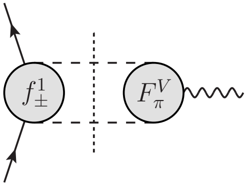


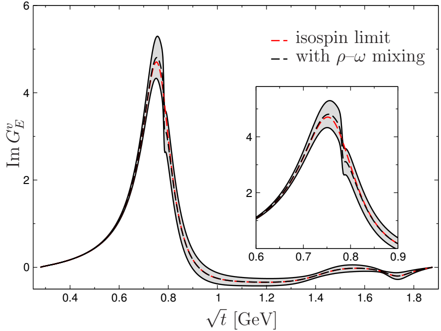
$$\begin{aligned}
\mathrm{Im}\, G_E^v(t) = {} & \frac{q_t^3}{m_N\sqrt{t}} |\Omega_1^1(t)||f_+^1(t)|\theta(t-t_\pi) \\
& \times \left(1 + \alpha t + \frac{\varepsilon\, t}{M_\omega^2 + iM_\omega\Gamma_\omega - t}\right) \\
& + \varepsilon\, \mathrm{Im}\left(\frac{t}{M_\omega^2 - iM_\omega\Gamma_\omega - t}\right) \\
& \times \frac{1}{\pi}\int_{t_\pi}^{\infty} dt' \frac{\frac{q_t'^3}{m_N\sqrt{t'}}|\Omega_1^1(t')||f_+^1(t')|}{t' - t - i\varepsilon},
\end{aligned}$$

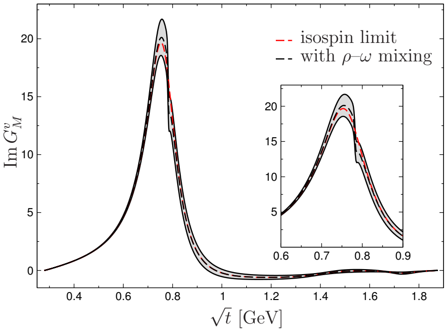
$$\begin{aligned}
\mathrm{Im}\, G_M^v(t) = {} & \frac{q_t^3}{\sqrt{2t}} |\Omega_1^1(t)||f_-^1(t)|\theta(t-t_\pi) \\
& \times \left(1 + \alpha t + \frac{\varepsilon\, t}{M_\omega^2 + iM_\omega\Gamma_\omega - t}\right) \\
& + \varepsilon\, \mathrm{Im}\left(\frac{t}{M_\omega^2 - iM_\omega\Gamma_\omega - t}\right) \\
& \times \frac{1}{\pi}\int_{t_\pi}^{\infty} dt' \frac{\frac{q_t'^3}{\sqrt{2t'}}|\Omega_1^1(t')||f_-^1(t')|}{t' - t - i\varepsilon}.
\end{aligned}$$

Nucleon form factor: $\pi\pi$ continuum

- $\pi\pi \to \bar{N}N$ partial waves + F_π^V pion form factor
  $\hookrightarrow$ $\pi\pi$ contribution to the isovector spectral functions
- consistent $\pi\pi$ phase shifts in $f_1^\pm$ and F_π^V
  $\hookrightarrow$ Watson theorem is satisfied
- modern pion form factor data [BaBar 2009, KLOE 2012, BESIII (2015)]
- isospin breaking: $m_p - m_n$ in pole terms, subthreshold parameters, consistent $\rho - \omega$ mixing







[Hoferichter, Kubis, JRE, Hammer, Meißner (2016)]

Nucleon form factors: sum rules and proton radius puzzle

- sum rules for the isovector radii: $$\langle r^2_{E/M}\rangle^v = \frac{6}{\pi}\int\limits_{4M_\pi^2}^{\Lambda} dt' \frac{\operatorname{Im} G^v_{E/M}(t')}{t'^2}$$

	$\Lambda = 1\,\text{GeV}$	$\Lambda = 2m_N$
$\langle r_E^2\rangle^v$ [fm^2]	0.418(32)	0.405(36)
$\langle r_M^2\rangle^v$ [fm^2]	1.83(10)	1.81(11)

- correcting normalization by single heavier resonance: ρ', ρ'':
  reduces the radii only to: $\Delta\langle r_E^2\rangle^v = -(0.006\ldots 0.008)\,\text{fm}^2$
  $$\Delta\langle r_M^2\rangle^v = -(0.05\ldots 0.07)\,\text{fm}^2$$
- with $\langle r_E^2\rangle^n = -0.1161(22)\,\text{fm}^2$ (n scattering on heavy atoms):
  $\hookrightarrow$ proton radius puzzle $\Longleftrightarrow$ isovector radius puzzle

$$\langle r_E^2\rangle^v = 0.412\,\text{fm}^2\ (\mu\text{H}) \quad \text{vs.} \quad \langle r_E^2\rangle^v = 0.442\,\text{fm}^2\ (\text{CODATA})$$

▷ mild preference for small proton charge radius

[Hoferichter, Kubis, JRE, Hammer, Meißner (2016)]

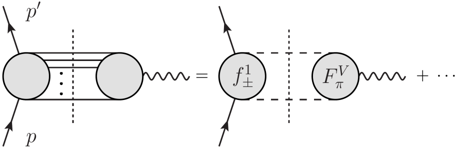
Nucleon form factors: relation to the antisymmetric quark tensor

- matrix elements $\langle N(p')|\bar{q}\sigma^{\mu\nu}q|N(p)\rangle$ input for BSM searches

  $\hookrightarrow$ responsible for χN and $\bar{\chi}N$ spin–dependent cross section difference

- isovector component: $J^{PC} = 1^{--}$ states related to P-wave $\pi\pi$ **spectral function**

  $\hookrightarrow$ **equivalence** of **vector** and **antisymmetric tensor** representations [Ecker et al. (1989)]



- in the elastic region:

$$F^{q,v}_{1,T}(t) = 0,$$

$$F^{q,v}_{2,T}(t) = -\frac{m_N}{2M_\pi} B^{\pi,q}_T(0)\big(F^v_1(t) + F^v_2(t)\big),$$

$$F^{q,v}_{3,T}(t) = \frac{m_N}{4M_\pi} B^{\pi,q}_T(0) F^v_2(t),$$

- combine with 1^{+-} and lattice normalizations to obtain full estimates for $\bar{q}\sigma^{\mu\nu}q$ matrix elements [Hoferichter, Kubis, JRE, Stoffer (2019)]

Chiral low-energy constants

	NLO	N^{2}LO	N^{3}LO
c_1 [GeV^{-1}]	-0.74 ± 0.02	-1.07 ± 0.02	-1.11 ± 0.03
c_2 [GeV^{-1}]	1.81 ± 0.03	3.20 ± 0.03	3.13 ± 0.03
c_3 [GeV^{-1}]	-3.61 ± 0.05	-5.32 ± 0.05	-5.61 ± 0.06
c_4 [GeV^{-1}]	2.17 ± 0.03	3.56 ± 0.03	4.26 ± 0.04
$\bar{d}_1 + \bar{d}_2$ [GeV^{-2}]	—	1.04 ± 0.06	7.42 ± 0.08
$\bar{d}_3$ [GeV^{-2}]	—	-0.48 ± 0.02	-10.46 ± 0.10
$\bar{d}_5$ [GeV^{-2}]	—	0.14 ± 0.05	0.59 ± 0.05
$\bar{d}_{14} - \bar{d}_{15}$ [GeV^{-2}]	—	-1.90 ± 0.06	-12.18 ± 0.12
$\bar{e}_{14}$ [GeV^{-3}]	—	—	0.89 ± 0.04
$\bar{e}_{15}$ [GeV^{-3}]	—	—	-0.97 ± 0.06
$\bar{e}_{16}$ [GeV^{-3}]	—	—	-2.61 ± 0.03
$\bar{e}_{17}$ [GeV^{-3}]	—	—	0.01 ± 0.06
$\bar{e}_{18}$ [GeV^{-3}]	—	—	-4.20 ± 0.05

- Subthreshold errors tiny, chiral expansion dominates uncertainty
- $\bar{d}_i$ at N^{3}LO increase by an order of magnitude
  - $\hookrightarrow$ due to terms proportional to $g_A^2(c_3 - c_4) = -16$ GeV^{-1}
  - $\hookrightarrow$ mimic loop diagrams with Δ degrees of freedom
- What's going on with chiral convergence?
  - $\hookrightarrow$ look at convergence of threshold parameters with LECs fixed at subthreshold point

Convergence of the chiral series

	NLO	N^2LO	N^3LO	RS
a^+_{0+} $[10^{-3}M_\pi^{-1}]$	-23.8	0.2	-7.9	-0.9 ± 1.4
a^-_{0+} $[10^{-3}M_\pi^{-1}]$	79.4	92.9	59.4	85.4 ± 0.9
a^+_{1+} $[10^{-3}M_\pi^{-3}]$	102.6	121.2	131.8	131.2 ± 1.7
a^-_{1+} $[10^{-3}M_\pi^{-3}]$	-65.2	-75.3	-89.0	-80.3 ± 1.1
a^+_{1-} $[10^{-3}M_\pi^{-3}]$	-45.0	-47.0	-72.7	-50.9 ± 1.9
a^-_{1-} $[10^{-3}M_\pi^{-3}]$	-11.2	-2.8	-22.6	-9.9 ± 1.2
b^+_{0+} $[10^{-3}M_\pi^{-3}]$	-70.4	-23.3	-44.9	-45.0 ± 1.0
b^-_{0+} $[10^{-3}M_\pi^{-3}]$	20.6	23.3	-64.7	4.9 ± 0.8

- N^3LO results bad due to large Delta loops
- matching to ChPT with the explicit Δ's

  $\hookrightarrow$ improvement of the chiral convergence [Siemens, JRE, Epelbaum, Hoferichter, Krebs, Kubis, Meißner (2016)]
- Conclusion: lessons for few-nucleon applications

  $\hookrightarrow$ either include the Δ to reduce the size of the loop corrections or use LECs from subthreshold kinematics

  $\hookrightarrow$ error estimates: consider chiral convergence of a given observable, difficult to assign a global chiral error to LECs

Chiral Low Energy Constants with Δ's

	HB-*NN*		HB-πN		covariant	
N^2LO	Q^3	ε^3	Q^3	ε^3	Q^3	ε^3
c_1	−1.08(2)	−1.25(3)	−1.08(2)	−1.24(3)	−1.00(2)	−1.19(4)
c_2	3.26(3)	1.71(1.01)	3.26(3)	1.13(1.02)	2.55(3)	1.14(19)
c_3	−5.39(5)	−2.68(84)	−5.39(5)	−2.75(84)	−4.90(5)	−2.56(40)
c_4	3.62(3)	1.57(16)	3.62(3)	1.58(16)	3.08(3)	1.33(20)
d_{1+2}	1.02(6)	0.14(17)	1.02(6)	−0.07(18)	1.78(6)	0.62(16)
d_3	−0.46(2)	−0.84(14)	−0.46(2)	−0.48(15)	−1.12(2)	−1.45(5)
d_5	0.15(5)	0.80(7)	0.15(5)	0.47(6)	−0.05(5)	0.29(6)
d_{14-15}	−1.85(6)	−1.09(30)	−1.85(6)	−0.72(31)	−2.27(6)	−0.98(13)
N^3LO	Q^4	ε^4	Q^4	ε^4	Q^4	ε^4
c_1	−1.11(3)	−1.11(3)	−1.11(3)	−1.11(3)	−1.12(3)	−1.10(3)
c_2	3.61(4)	1.41(38)	3.17(3)	1.28(20)	3.35(3)	1.16(20)
c_3	−5.60(6)	−1.88(45)	−5.67(6)	−2.04(39)	−5.70(6)	−2.10(39)
c_4	4.26(4)	2.03(28)	4.35(4)	2.07(29)	3.97(3)	1.91(27)
d_{1+2}	6.37(9)	1.78(31)	7.66(9)	2.90(30)	4.70(7)	1.78(24)
d_3	−9.18(9)	−3.64(36)	−10.77(10)	−5.91(50)	−5.26(5)	−3.25(14)
d_5	0.87(5)	1.52(7)	0.59(5)	1.03(7)	0.31(5)	0.66(6)
d_{14-15}	−12.56(12)	−4.38(54)	−13.44(12)	−5.17(55)	−8.84(10)	−3.41(41)
e_{14}	1.16(4)	1.64(10)	0.85(4)	1.12(16)	1.17(4)	1.28(11)
e_{15}	−2.26(6)	−4.95(15)	−0.83(6)	−3.30(25)	−2.58(7)	−3.07(13)
e_{16}	−0.29(3)	4.21(16)	−2.75(3)	1.92(43)	−1.77(3)	1.71(17)
e_{17}	−0.17(6)	−0.44(6)	0.03(6)	−0.39(7)	−0.45(6)	−0.51(7)
e_{18}	−3.47(5)	1.34(29)	−4.48(5)	0.67(31)	−1.68(5)	1.30(17)

Threshold kinematics from subthreshold with Δ's

	HB-NN		HB-πN		covariant		RS
N^2LO	Q^3	ϵ^3	Q^3	ϵ^3	Q^3	ϵ^3	
a^+_{0+} $[10^{-3}M_\pi^{-1}]$	0.5	−9.8(10.9)	0.5	−0.4(9.2)	−14.8	1.0(17.3)	−0.9(1.4)
a^-_{0+} $[10^{-3}M_\pi^{-1}]$	92.2	92.7(1.0)	92.9	90.5(9)	89.9	81.7(1.6)	85.4(9)
a^+_{1+} $[10^{-3}M_\pi^{-3}]$	113.8	125.8(16.7)	121.7	127.2(18.4)	116.4	128.5(9.6)	131.2(1.7)
a^-_{1+} $[10^{-3}M_\pi^{-3}]$	−74.8	−77.4(2.5)	−75.5	−78.4(2.6)	−75.1	−79.7(3.0)	−80.3(1.1)
a^+_{1-} $[10^{-3}M_\pi^{-3}]$	−54.1	−53.4(14.1)	−47.0	−52.5(15.8)	−55.5	−52.5(8.5)	−50.9(1.9)
a^-_{1-} $[10^{-3}M_\pi^{-3}]$	−14.1	−13.1(2.7)	−2.5	−7.8(3.0)	−10.4	−9.7(4.1)	−9.9(1.2)
b^+_{0+} $[10^{-3}M_\pi^{-3}]$	−45.7	−38.1(9.6)	−22.1	−23.7(14.4)	−50.9	−34.7(12.1)	−45.0(1.0)
b^-_{0+} $[10^{-3}M_\pi^{-3}]$	35.9	26.4(1.0)	22.6	17.6(8)	21.6	14.2(2.0)	4.9(8)
N^3LO	Q^4	ϵ^4	Q^4	ϵ^4	Q^4	ϵ^4	
a^+_{0+} $[10^{-3}M_\pi^{-1}]$	−1.5	−1.5(8.5)	−8.0	1.2(20.4)	−5.7	−0.8(10.3)	−0.9(1.4)
a^-_{0+} $[10^{-3}M_\pi^{-1}]$	68.5	96.3(2.0)	58.6	70.0(3.3)	83.8	83.6(1.9)	85.4(9)
a^+_{1+} $[10^{-3}M_\pi^{-3}]$	134.3	136.0(9.7)	132.1	135.2(8.7)	128.0	132.7(9.0)	131.2(1.7)
a^-_{1+} $[10^{-3}M_\pi^{-3}]$	−80.9	−80.0(3.4)	−90.1	−86.4(2.7)	−78.1	−81.1(3.6)	−80.3(1.1)
a^+_{1-} $[10^{-3}M_\pi^{-3}]$	−55.7	−47.5(10.5)	−73.7	−56.9(7.1)	−53.5	−51.4(7.9)	−50.9(1.9)
a^-_{1-} $[10^{-3}M_\pi^{-3}]$	−10.0	−5.6(4.9)	−23.7	−14.4(6.5)	−11.8	−10.4(5.7)	−9.9(1.2)
b^+_{0+} $[10^{-3}M_\pi^{-3}]$	−42.2	−31.4(8.1)	−44.5	−32.6(21.3)	−54.7	−33.9(8.5)	−45.0(1.0)
b^-_{0+} $[10^{-3}M_\pi^{-3}]$	−31.6	7.1(2.3)	−65.2	−34.1(5.7)	2.3	2.9(2.1)	4.9(8)

Nucleon resonances: conventions for the $\pi\pi$ form factors

- starting from the $\pi\pi$ unitarity relation

$$\text{Im}\, t^I_J(t) = \sigma_\pi(t)\big|t^I_J(t)\big|^2$$

- defining the $\pi\pi$ residues as

$$t^0_{0,\text{II}}(t) = \frac{g^2_{S\pi\pi}}{16\pi(t_S - t)}, \qquad t^1_{1,\text{II}}(t) = \frac{g^2_{\rho\pi\pi}(t - t_\pi)}{48\pi(t_\rho - t)}$$

with $S = f_0(500)$ or $f_0(980)$

- taking into account the unitarity relations for the scalar and vector form factors of the pion

$$\text{Im}\, F^\theta_\pi(t) = \sigma_\pi(t)\big[t^0_0(t)\big]^* F^\theta_\pi(t), \quad \text{Im}\, F^V_\pi(t) = \sigma_\pi(t)\big[t^1_1(t)\big]^* F^V_\pi(t)$$

- and writing the form factors on the second sheet as

$$F^\theta_{\pi,\text{II}}(t) = \sqrt{\frac{2}{3}}\frac{F_S g_{S\pi\pi} t_S}{t_S - t}, \quad F^V_{\pi,\text{II}}(t) = \frac{g_{\rho\pi\pi}}{g_{\rho\gamma}}\frac{t_\rho}{t_\rho - t}$$

- one has

$$\frac{F_S}{g_{S\pi\pi}} = i\sqrt{\frac{3}{2}}\frac{\sigma_\pi(t_S)}{8\pi t_S}F^\theta_{\pi,\text{I}}(t_S), \qquad \frac{1}{g_{\rho\gamma}g_{\rho\pi\pi}} = i\frac{\sigma^3_\pi(t_\rho)}{24\pi}F^V_{\pi,\text{I}}(t_\rho)$$

Nucleon resonances: conventions for t-channel amplitudes

- in the same way, starting from the unitarity relations

$$\text{Im}\, f_{\pm}^J(t) = \sigma_\pi(t)\,[t_J^I(t)]^*\, f_{\pm}^J(t)$$

- defining the nucleon residues as

$$f^0_{+,\text{II}}(t) = -\frac{p_t^2}{2\pi\sqrt{6}}\,\frac{g_{SNN}\,g_{S\pi\pi}}{t_S - t}\,,\quad f^1_{+,\text{II}}(t) = \frac{m_N g_{\rho\pi\pi}}{12\pi}\,\frac{g^{(1)}_{\rho NN} + \frac{t_\rho}{4m_N^2}\,g^{(2)}_{\rho NN}}{t_\rho - t}\,,\quad f^1_{-,\text{II}}(t) = \frac{\sqrt{2}\,g_{\rho\pi\pi}}{12\pi}\,\frac{g^{(1)}_{\rho NN} + g^{(2)}_{\rho NN}}{t_\rho - t}$$

- taking the unitarity relations for the scalar and vector form factor of the nucleon

$$\text{Im}\,\theta_N(t) = \frac{\sigma_\pi(t)}{4m_N^2 - t}\,\frac{3}{2}\,[F_\pi^\theta(t)]^*\, f_+^0(t),\quad \text{Im}\, G_E^v(t) = \frac{\sigma_\pi(t) q_t^2}{2m_N}\,[F_\pi^V(t)]^*\, f_+^1(t),\quad \text{Im}\, G_M^v(t) = \frac{\sigma_\pi(t) q_t^2}{2\sqrt{2}}\,[F_\pi^V(t)]^*\, f_-^1(t)$$

- one obtains for the scalar channel

$$\frac{g_{SNN}}{g_{S\pi\pi}} = i\sqrt{6}\,\frac{\sigma_\pi(t_S)}{4m_N^2 - t_S}\, f^0_{+,\text{I}}(t_S),$$

- and for the vector

$$\frac{g^{(1)}_{\rho NN}}{g_{\rho\pi\pi}} = -2i\sigma_\pi(t_\rho)\,\frac{m_N q_t^2}{p_t^2}\left[f^1_{+,\text{I}}(t_\rho) - \frac{t_\rho}{4\sqrt{2}\,m_N}\, f^1_{-,\text{I}}(t_\rho)\right]$$

$$\frac{g^{(2)}_{\rho NN}}{g_{\rho\pi\pi}} = 2i\sigma_\pi(t_\rho)\,\frac{m_N q_t^2}{p_t^2}\left[f^1_{+,\text{I}}(t_\rho) - \frac{m_N}{\sqrt{2}}\, f^1_{-,\text{I}}(t_\rho)\right]$$