Baryon-meson scattering amplitude in the $1/N_c$ expansion of QCD

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August 27th, 2024

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- **QCD** is a gauge theory with the local symmetry group $SU(N_c)$, with $N_c = 3$ color charges.

- Analytical computation of hadron properties from first principles are not allowed due QCD is strongly coupled at low energies.

- Two relevant theories appear as response to the absence of a perturbative parameter:
  1. Chiral Perturbation Theory (ChPT).
  2. $1/N_c$ expansion of QCD.

- *Baryon-meson scattering* is a fundamental nuclear physics process which can be analyzed within both formalisms.

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- The $1/N_c$ *expansion for QCD* provides us of a systematic method considering the *large* N_c limit.
- In this limit, the contracted spin flavor symmetry $SU(2N_f)$,where N_f is the number of flavors, emerges with the spin-flavor algebra, with generators

$$J^i = q^\dagger \left(\frac{\sigma^i}{2} \otimes \mathbb{1} \right) q, \quad T^a = q^\dagger \left(\mathbb{1} \otimes \frac{\lambda^a}{2} \right) q, \quad G^{ia} = q^\dagger \left(\frac{\sigma^i}{2} \otimes \frac{\lambda^a}{2} \right) q,$$

that satisfy the algebraic structure given by

$$\left[J^i, T^a\right] = 0,$$
$$\left[J^i, J^j\right] = i\epsilon^{ijk} J^k, \quad \left[T^a, T^b\right] = i f^{abc} T^c,$$
$$\left[J^i, G^{ja}\right] = i\epsilon^{ijk} G^{ka}, \quad \left[T^a, G^{ib}\right] = i f^{abc} G^{ic},$$
$$\left[G^{ia}, G^{jb}\right] = \tfrac{i}{4}\delta^{ij} f^{abc} T^c + \tfrac{i}{2N_f}\delta^{ab}\epsilon^{ijk} J^k + \tfrac{i}{2}\epsilon^{ijk} d^{abc} G^{kc}.$$

$1/N_c$ Expansion for QCD

Using the spin flavor symmetry, it is possible to expand any QCD operator as

$$\mathcal{O}_{QCD} = \sum_{n=0}^{N_c} \frac{1}{{N_c}^n} \mathcal{O}_n \left(J^i, T^a, G^{ia} \right).$$

where $\mathcal{O}_n$ are called n-*body* operators.

The n-body operators are combinations of 1-body operators ($SU(2N_f)$ generators), while the unique 0-body operator is the identity $\mathbb{1}$.

- 2-body

$$\mathcal{O}_2^{ia} = \epsilon^{ijk}\{J^j, G^{ka}\} = i[J^2, G^{ia}], \qquad \mathcal{D}_2^{ia} = J^i T^a.$$

- 3-body

$$\mathcal{O}_3^{ia} = \{J^2, G^{ia}\} - \frac{1}{2}\{J^i, \{J^j, G^{ja}\}\}, \qquad \mathcal{D}_3^{ia} = \{J^i, \{J^j, G^{ja}\}\},$$

- n-body operators

$$\mathcal{O}_{n+2}^{ia} = \{J^2, \mathcal{O}_n^{ia}\}, \qquad \mathcal{D}_{n+2}^{ia} = \{J^2, \mathcal{D}_n^{ia}\}.$$

$SU(N)$ Projection Operators

- Since symmetry transformation properties for each operator are important, a useful tool are $SU(N)$ **projection operators**.
- Projection operators act on the n indices of a tensor operators that transform under $\prod_{i=1}^{n} adj\otimes$, and obtain a particular irrep from the decomposition. The projectors are given by

$$\boxed{\mathcal{P}^{(m)} = \prod_{i=1}^{k} \left[\frac{C - c_{n_i}}{c_m - c_{n_i}} \right], \qquad c_m \neq c_{n_i},}$$

  where k labels all the eigenvalues for the quadratic Casimir C and c_m are the respective eigenvalues.
- The application of projection operators over a $SU(N)$ tensor operator $\prod_{i=1}^{n} Q_i^{a_i}$ follows:

$$\mathcal{P}^{(m)} \prod_{i=1}^{n} Q_i^{a_i} = \tilde{Q}^{a_1 \ldots a_n}.$$

Brief example: Projection operators for $SU(2)$

- $SU(2)$ is the simplest non-Abelian group, and its realizations describe spin and isospin.
- The spin-1 objects transforms under the adjoint representation given by **3**. So, for two spin indices objects the representation is $adj \otimes adj$, then

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5}.$$

- The projection operators are:

$$\left[\mathcal{P}^{(0)}\right]^{a_1 a_2 b_1 b_2} = \frac{1}{3}\delta^{a_1 a_2}\delta^{b_1 b_2},$$

$$\left[\mathcal{P}^{(1)}\right]^{a_1 a_2 b_1 b_2} = \frac{1}{2}\left(\delta^{a_1 b_1}\delta^{a_2 b_2} - \delta^{a_2 b_1}\delta^{a_1 b_2}\right),$$

$$\left[\mathcal{P}^{(2)}\right]^{a_1 a_2 b_1 b_2} = \frac{1}{2}\left(\delta^{a_1 b_1}\delta^{a_2 b_2} + \delta^{a_2 b_1}\delta^{a_1 b_2}\right) - \frac{1}{3}\delta^{a_1 a_2}\delta^{b_1 b_2}.$$

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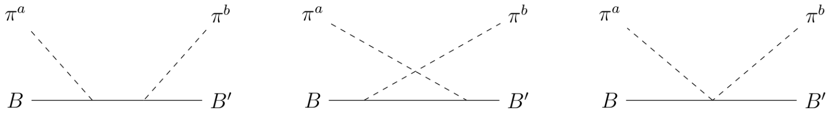
Baryon-Meson Scattering at Tree Level

The baryon-meson scattering process $B + \pi^a \longrightarrow B' + \pi^b$ is described at tree level by the diagrams:



For the first two diagrams the amplitude is given by:

$$A^{ab}_{\text{tree}} = -\frac{1}{f^2} k^i k'^j \left[\frac{1}{k^0}[A^{jb}, A^{ia}] + \frac{1}{{k^0}^2}[A^{jb}, [\mathcal{M}, A^{ia}]] + \frac{1}{{k^0}^3}[A^{jb}, [\mathcal{M}, [\mathcal{M}, A^{ia}]]] \ldots \right],$$

where $\mathcal{M}$ is the baryon mass operator, A^{ia} is the baryon axial vector current and $f \approx 93$ MeV is the pion decay constant.

The baryon operator A^{ab}_{tree} is a spin-zero object with two adjoint (octet) indices. The tensor product of two adjoint representations $\mathbf{8} \otimes \mathbf{8}$ can decomposed as

$$(\mathbf{8} \otimes \mathbf{8})_S = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{27},$$
$$(\mathbf{8} \otimes \mathbf{8})_A = \mathbf{8} \oplus \mathbf{10} \oplus \overline{\mathbf{10}}.$$

Baryon-Meson Scattering Amplitude at Tree Level

The matrix elements of A^{ab}_{tree} describe the scattering amplitude as

$$\mathcal{A}_{\text{tree}}(B+\pi^a \longrightarrow B'+\pi^b) \equiv \langle B'\pi^b|A^{ab}_{\text{tree}}|B\pi^a\rangle,$$

and its expansion up to 7-body operators is given by

$$\mathcal{A}_{\text{tree}}(B+\pi^a \longrightarrow B'+\pi^b) = -\frac{1}{f^2 k_0}\sum_{m=1}^{139}\left(C_m^{(S)}+C_m^{(A)}\right)\langle B'\pi^b|S_m^{(ij)(ab)}|B\pi^a\rangle$$

- $S_m^{(ij)(ab)}$ constitute a ‘complete’ linearly independent basis for spin-2 baryon operators with two adjoint flavor indices.
- The C_m coefficients are come along with the symmetric and antisymmetric pieces of A^{ab} from the projection operators.

Example: $N\pi \to N\pi$ scattering processes

$$
\begin{aligned}
&f^2 k^0 \mathcal{A}_{\text{tree}}(p+\pi^+ \to p+\pi^+) \\
&= \left[-\frac{25}{72}a_1^2 - \frac{5}{36}a_1 b_2 - \frac{25}{108}a_1 b_3 - \frac{1}{72}b_2^2 - \frac{5}{108}b_2 b_3 - \frac{25}{648}b_3^2 \right. \\
&\quad + \frac{2}{9}\left[1 - \frac{2\Delta}{k^0} + \frac{\Delta^2}{k^{0\,2}}\right]\left[a_1^2 + a_1 c_3 + \frac{1}{4}c_3^2\right]\Bigg] \mathbf{k}\cdot\mathbf{k}' \\
&\quad + \left[\frac{25}{72}a_1^2 + \frac{5}{36}a_1 b_2 + \frac{25}{108}a_1 b_3 + \frac{1}{72}b_2^2 + \frac{5}{108}b_2 b_3 + \frac{25}{648}b_3^2 \right. \\
&\quad - \frac{2}{9}\left[1 - \frac{1}{2}\frac{\Delta}{k^0} + \frac{\Delta^2}{k^{0\,2}}\right]\left[a_1^2 + a_1 c_3 + \frac{1}{4}c_3^2\right]\Bigg] i(\mathbf{k}\times\mathbf{k}')^3 + \mathcal{O}\left[\frac{\Delta^3}{k^{0\,3}}\right] \\
&= f^2 k^0 \mathcal{A}_{\text{tree}}(n+\pi^- \to n+\pi^-),
\end{aligned}
$$

$$
\begin{aligned}
f^2 k^0 \mathcal{A}_{\text{tree}}(p+\pi^+ \to p+\pi^+) &= \left[-\frac{1}{2}(D+F)^2 + \frac{1}{9}\left[-\frac{k^0}{k^0-\Delta} + 3\frac{k^0}{k^0+\Delta}\right]\mathcal{C}^2\right]\mathbf{k}\cdot\mathbf{k}' \\
&\quad + \left[\frac{1}{2}(D+F)^2 - \frac{1}{18}\left[\frac{k^0}{k^0-\Delta} + 3\frac{k^0}{k^0+\Delta}\right]\mathcal{C}^2\right] i(\mathbf{k}\times\mathbf{k}')^3 \\
&= f^2 k^0 \mathcal{A}_{\text{tree}}(n+\pi^- \to n+\pi^-).
\end{aligned}
$$

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First-order SB effects are computed from the tensor product of the scattering amplitude itself, which transforms under $SU(2) \times SU(3)$ as $(2, \mathbf{8} \otimes \mathbf{8})$, and the perturbation, which transforms as $(0, \mathbf{8})$. The representations product decompose as

$$\mathbf{8} \otimes \mathbf{8} \otimes \mathbf{8} = 2(\mathbf{1}) \oplus 8(\mathbf{8}) \oplus 4(\mathbf{10} \oplus \overline{\mathbf{10}}) \oplus 6(\mathbf{27}) \oplus 2(\mathbf{35} \oplus \overline{\mathbf{35}}) \oplus \mathbf{64}.$$

The contribution for scattering amplitude from first-order SB $\delta\mathcal{A}$, can be organized as

$$\begin{aligned}
&f^2 k^0 \delta\mathcal{A}(B + \pi^a \to B' + \pi^b) = \\
&\sum_m \Big[N_c g_1^{(m)} k^i k'^j \langle B' \pi^b | [\mathcal{P}^{(m)} R_1^{(ij)}]^{(ab8)} | B \pi^a \rangle + N_c g_2^{(m)} k^i k'^j \langle B' \pi^b | [\mathcal{P}^{(m)} R_2^{(ij)}]^{(ab8)} | B \pi^a \rangle \\
&+ \sum_{r=3}^{16} g_r^{(m)} k^i k'^j \langle B' \pi^b | [\mathcal{P}^{(m)} R_r^{(ij)}]^{(ab8)} | B \pi^a \rangle + \frac{1}{N_c} \sum_{r=17}^{71} g_r^{(m)} k^i k'^j \langle B' \pi^b | [\mathcal{P}^{(m)} R_r^{(ij)}]^{(ab8)} | B \pi^a \rangle \\
&+ \frac{1}{N_c^2} \sum_{r=72}^{170} g_r^{(m)} k^i k'^j \langle B' \pi^b | [\mathcal{P}^{(m)} R_r^{(ij)}]^{(ab8)} | B \pi^a \rangle \Big].
\end{aligned}$$

First-order SB effects to Scattering Amplitudes for $N+\pi \to N+\pi$

$$
\begin{aligned}
f^2 k^0 \delta \mathbf{A}(p+\pi^+ \to p+\pi^+) &= (d_1^{(1)} + d_1^{(8)} + d_1^{(10+\overline{10})} + d_1^{(27)})\mathbf{k}\cdot\mathbf{k}' \\
&\quad + (e_1^{(1)} + e_1^{(8)} + e_1^{(10+\overline{10})} + e_1^{(27)}) i(\mathbf{k}\times\mathbf{k}')^3 \\
&= f^2 k^0 \delta \mathbf{A}(n+\pi^- \to n+\pi^-),
\end{aligned}
$$

$$
\begin{aligned}
f^2 k^0 \delta \mathbf{A}(p+\pi^- \to p+\pi^-) &= (d_1^{(1)} + d_1^{(8)} - d_1^{(10+\overline{10})} - d_1^{(27)} + d_2^{(8)} + d_2^{(27)})\mathbf{k}\cdot\mathbf{k}' \\
&\quad + (e_1^{(1)} + e_1^{(8)} - e_1^{(10+\overline{10})} + e_2^{(8)}) i(\mathbf{k}\times\mathbf{k}')^3 \\
&= f^2 k^0 \delta \mathbf{A}(n+\pi^+ \to n+\pi^+),
\end{aligned}
$$

$$
\begin{aligned}
f^2 k^0 \delta \mathbf{A}(p+\pi^0 \to p+\pi^0) &= \frac{1}{2}(2d_1^{(1)} + 2d_1^{(8)} + d_2^{(8)} + d_2^{(27)})\mathbf{k}\cdot\mathbf{k}' \\
&\quad + \frac{1}{2}(2e_1^{(1)} + 2e_1^{(8)} + e_1^{(27)} + e_2^{(8)}) i(\mathbf{k}\times\mathbf{k}')^3 \\
&= f^2 k^0 \delta \mathbf{A}(n+\pi^0 \to n+\pi^0),
\end{aligned}
$$

$$
\begin{aligned}
\sqrt{2} f^2 k^0 \delta \mathbf{A}(p+\pi^- \to n+\pi^0) &= (2d_1^{(10+\overline{10})} + 2d_1^{(27)} - d_2^{(8)} - d_2^{(27)})\mathbf{k}\cdot\mathbf{k}' \\
&\quad + (2e_1^{(10+\overline{10})} + e_1^{(27)} - e_2^{(8)}) i(\mathbf{k}\times\mathbf{k}')^3 \\
&= \sqrt{2} f^2 k^0 \delta \mathbf{A}(n+\pi^+ \to p+\pi^0),
\end{aligned}
$$

Contents

- A tree level analysis for the baryon-meson scattering processes in the context of the $1/N_c$ expansion have been developed.

- The projection operators have shown a strong applicability for a more systematic expansion by classifying transformation properties of the operators involved in the expansion.

- A first-order symmetry breaking analysis has been performed, but it still represents a challenge due the computational problem.

Thanks for your attention!