Three-hadron dynamics from lattice QCD

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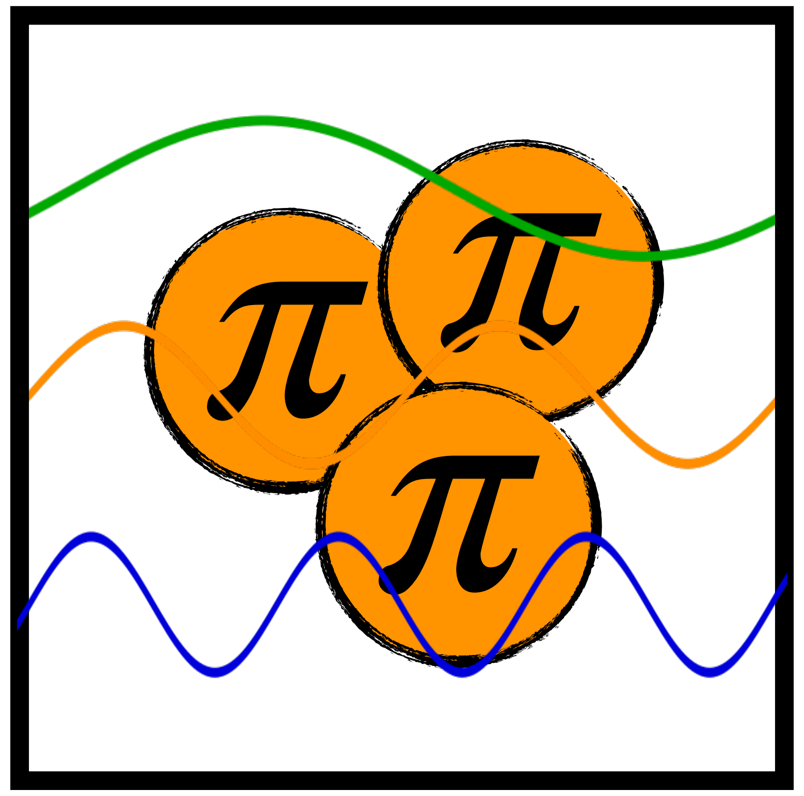
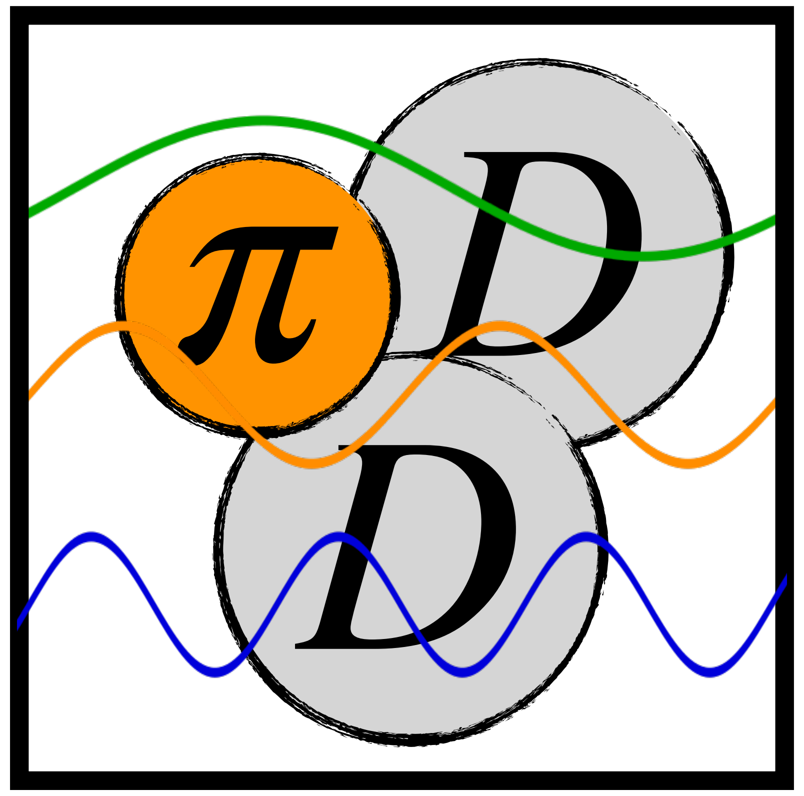
CD24, 27th August





The three-hadron frontier

Many reasons to study the three-hadron problem from lattice QCD

- ○ Resonances that decay to three (or more) hadrons
  - **Exotics:** $T_{cc} \to DD^*, DD\pi$
  - 3π **resonances:** $\omega(782),\ h_1(1170),\ a_1(1260)\ldots$
  - **Roper:** $N(1440) \to \Delta\pi \to N\pi\pi$

- ○ Interest in three-baryon forces: $NNN,\ NNY$

- ○ Electroweak processes $K \to 3\pi, \quad K^0 \leftrightarrow 3\pi \leftrightarrow \overline{K}^0, \quad \gamma \to 3\pi$

- ☑ Major developments in the three-particle finite-volume formalism

  [Hansen, Sharpe, PRD 2014 & 2015], [Hammer, Pang, Rusetsky, JHEP 2017] x 2
  [Mai, Döring, EPJA 2017] [...]

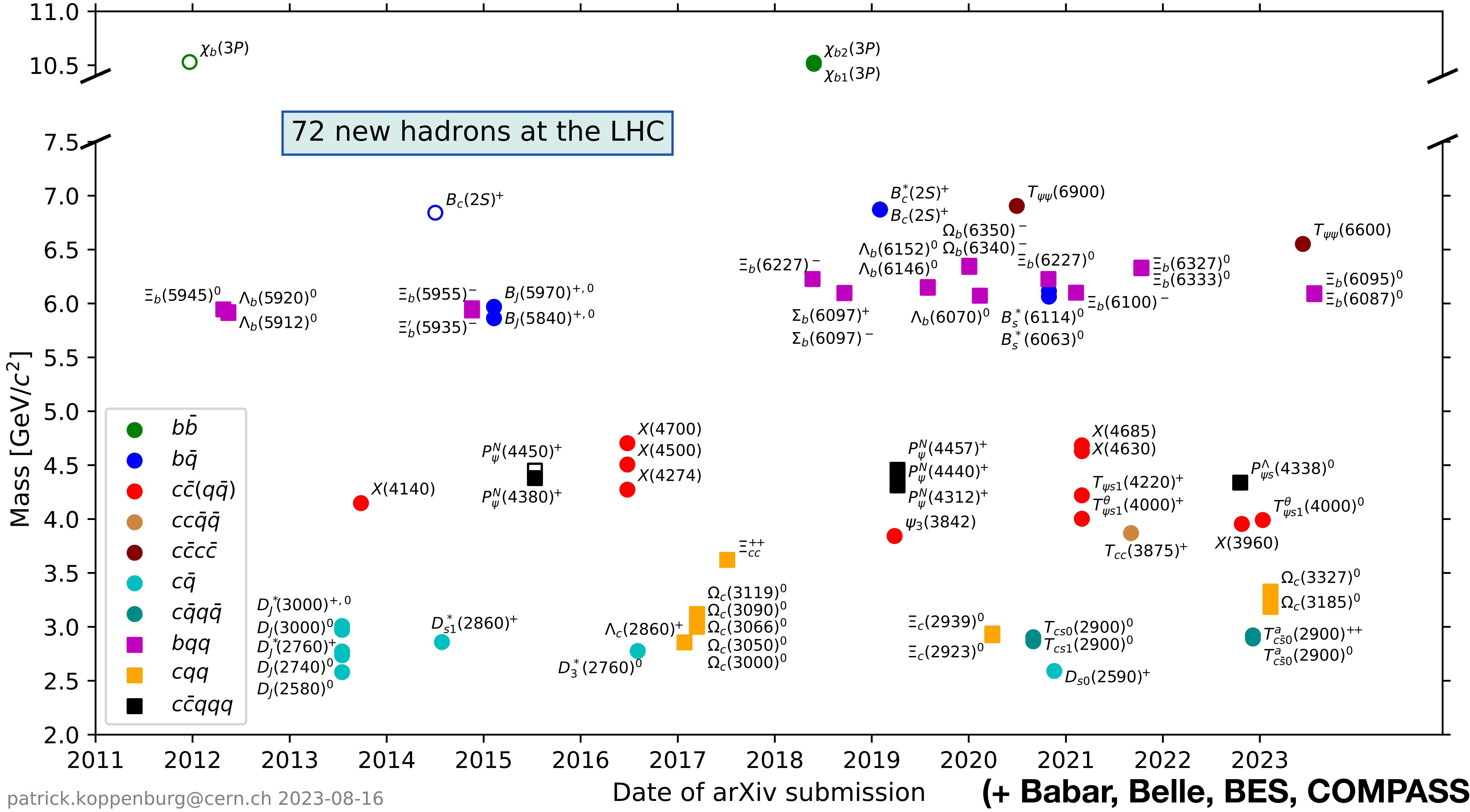
- ☑ See other related talks at this conference:

  [M. Hansen (plenary, Tuesday)]
  [F. Pittler (parallel, Monday), M. Sjö, A. Rusetsky, L. Meng, S. Dawid (parallel, Tuesday)]

Exotics and more

A growing hadron spectrum still requires first principles understanding

[https://www.nikhef.nl/~pkoppenb/particles.html]



(+ Babar, Belle, BES, COMPASS)

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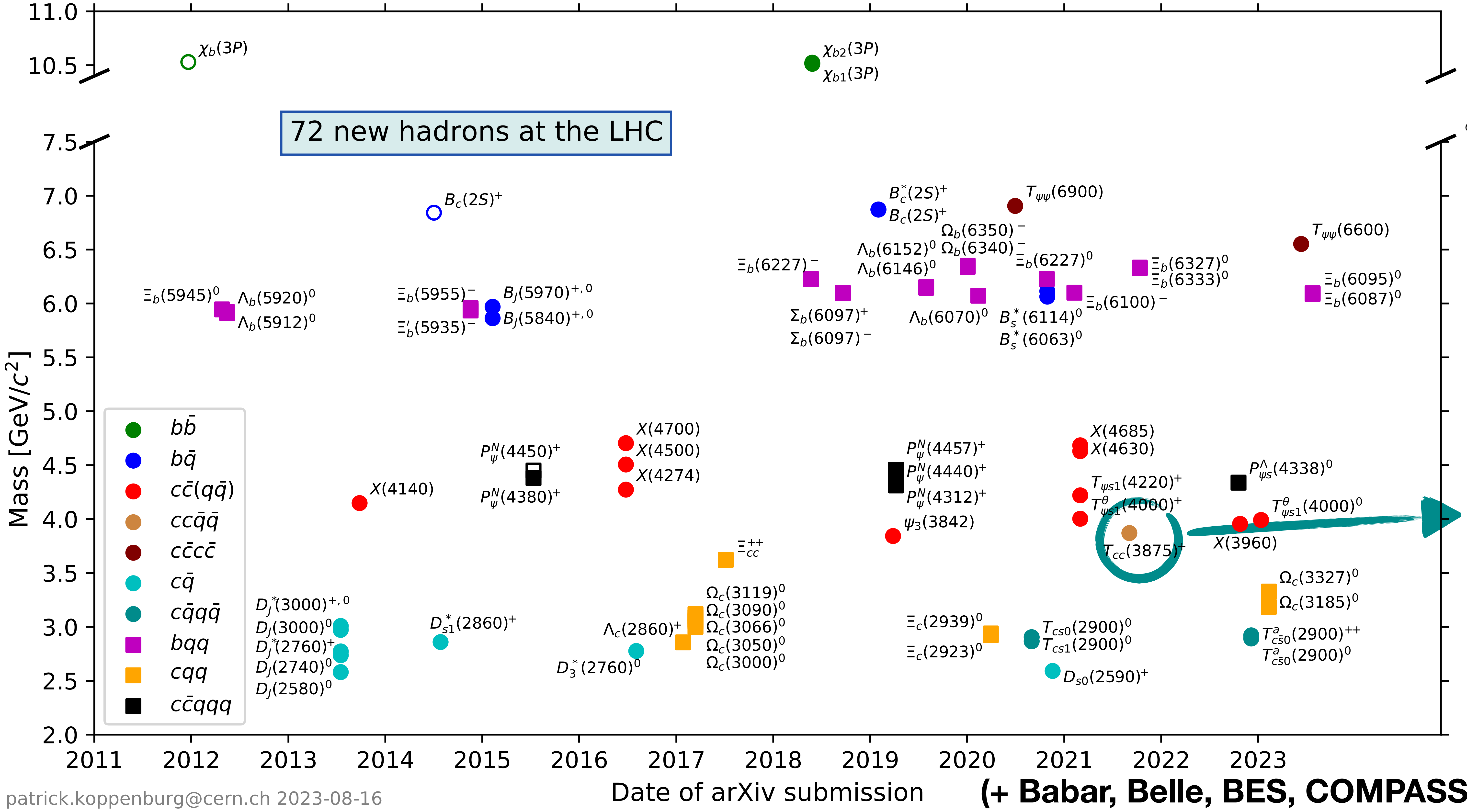

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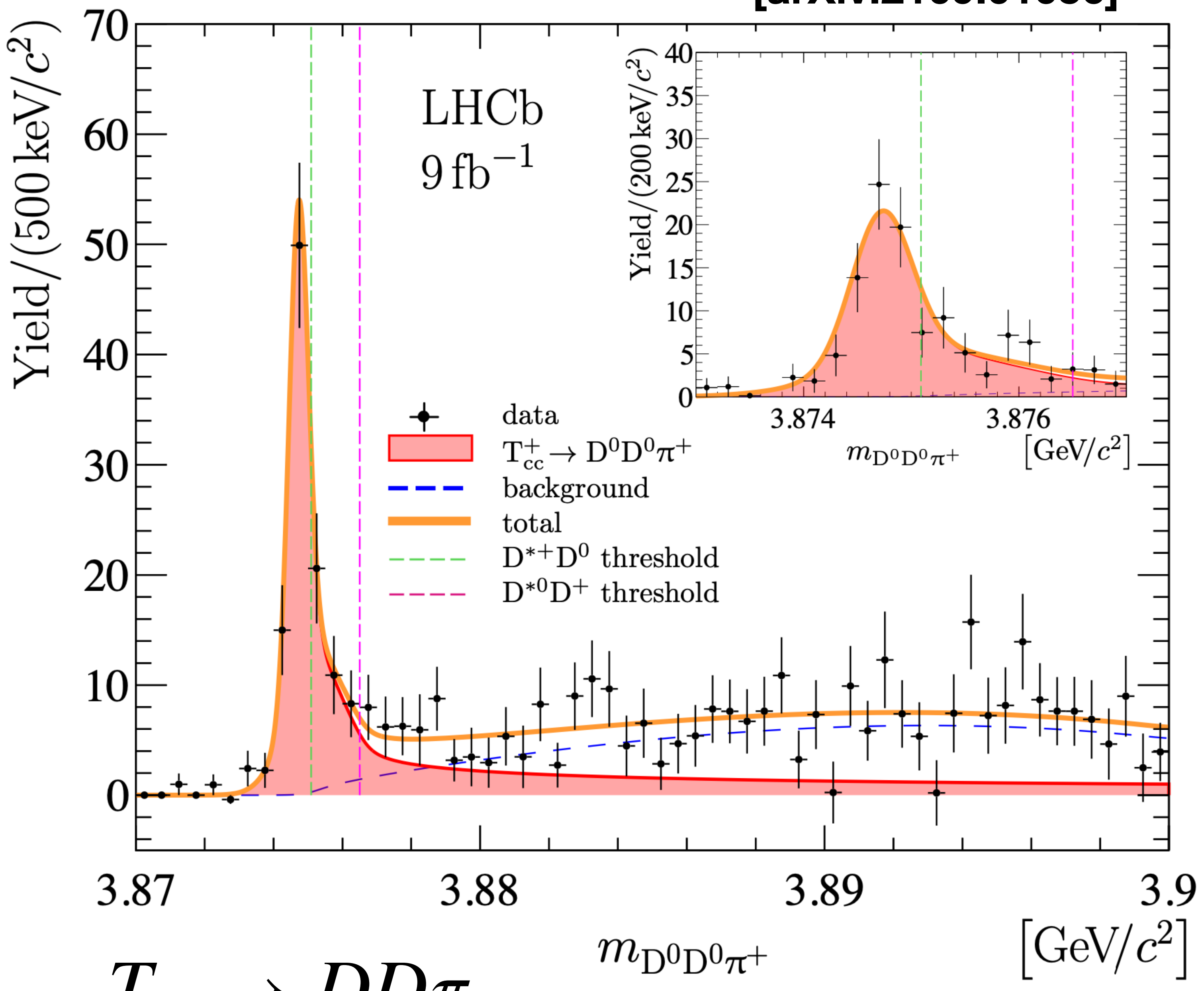
[arXiv:2109.01056]



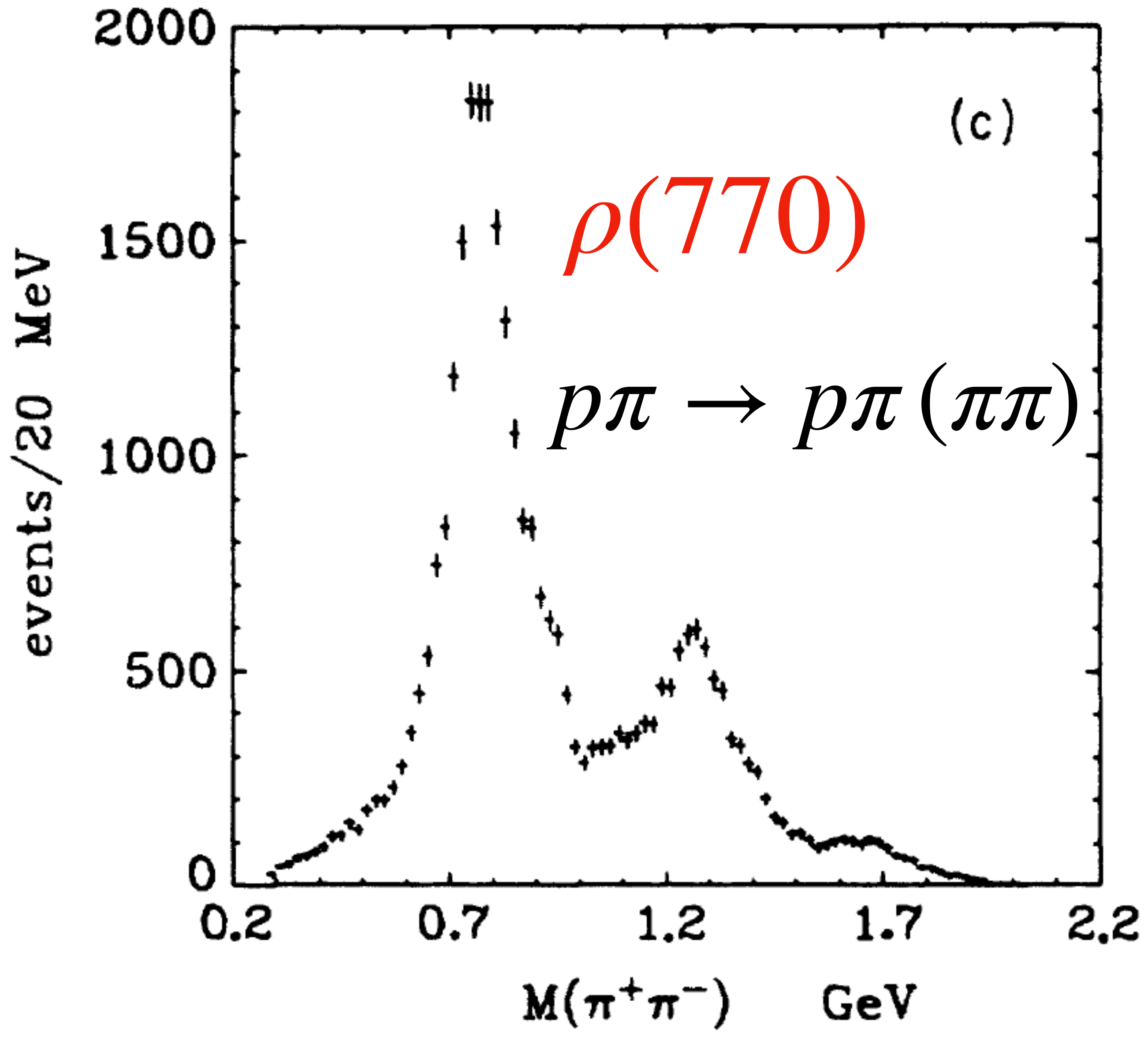
$T_{cc} \to DD\pi$

[M-L. Du, D. Wilson, M. Abolnikov (parallel, Monday)]
[S. Dawid (parallel, Tuesday)]

Infinite vs finite volume

Experiments

- Asymptotic states
- Direct access to cross sections



[Protopopescu et al, PRD7 1973]

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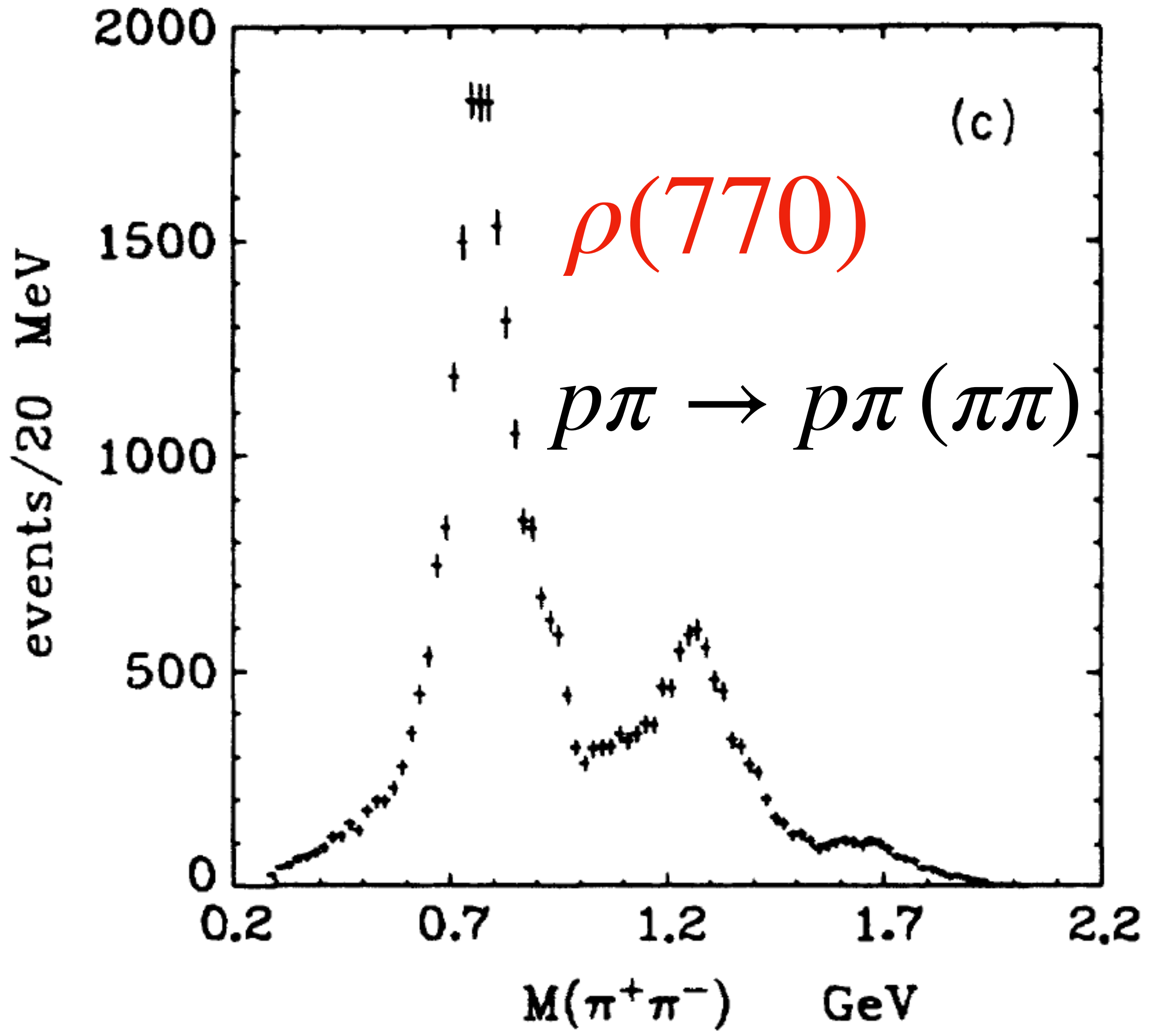
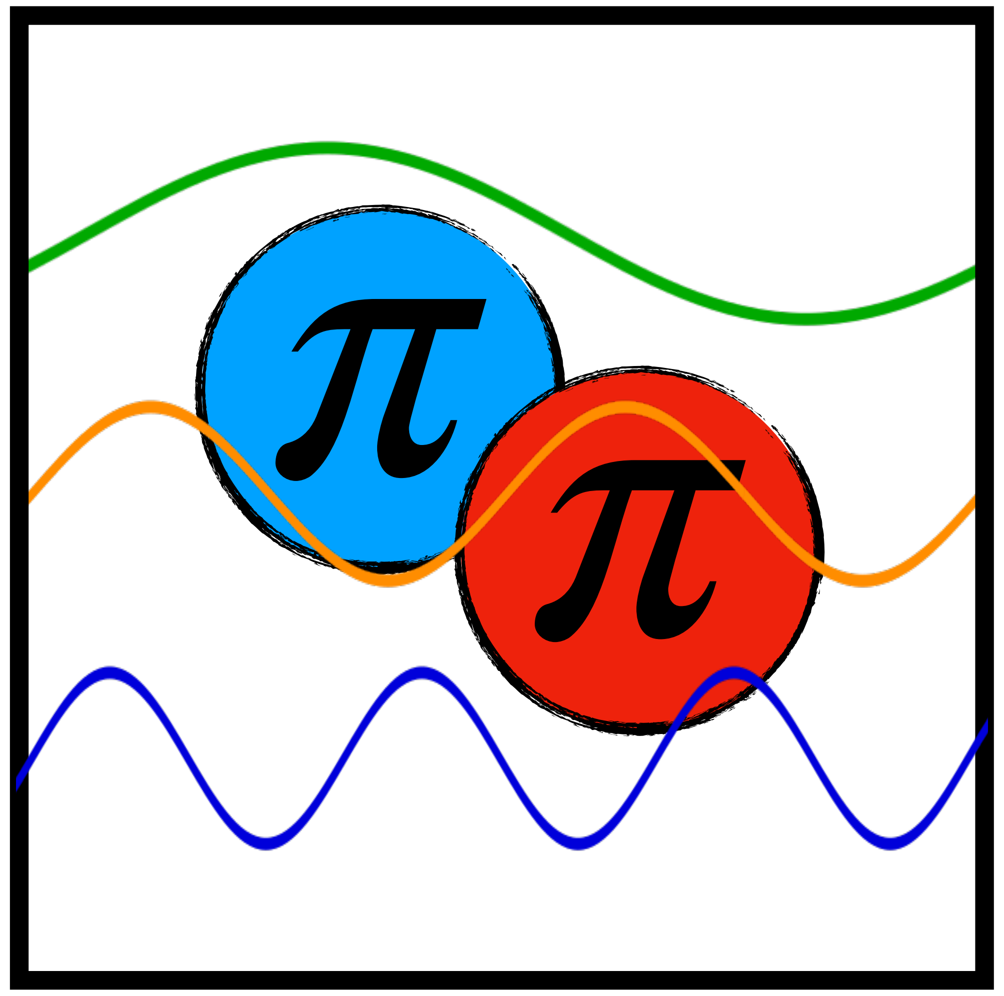


[Protopopescu et al, PRD7 1973]

Lattice QCD

- Euclidean time
- Stationary states in a box

$$C(t) = \langle \mathcal{O}(t)\mathcal{O}^\dagger(0)\rangle = \sum_n \left|\langle 0|\mathcal{O}(0)|n\rangle\right|^2 e^{-E_n t}$$



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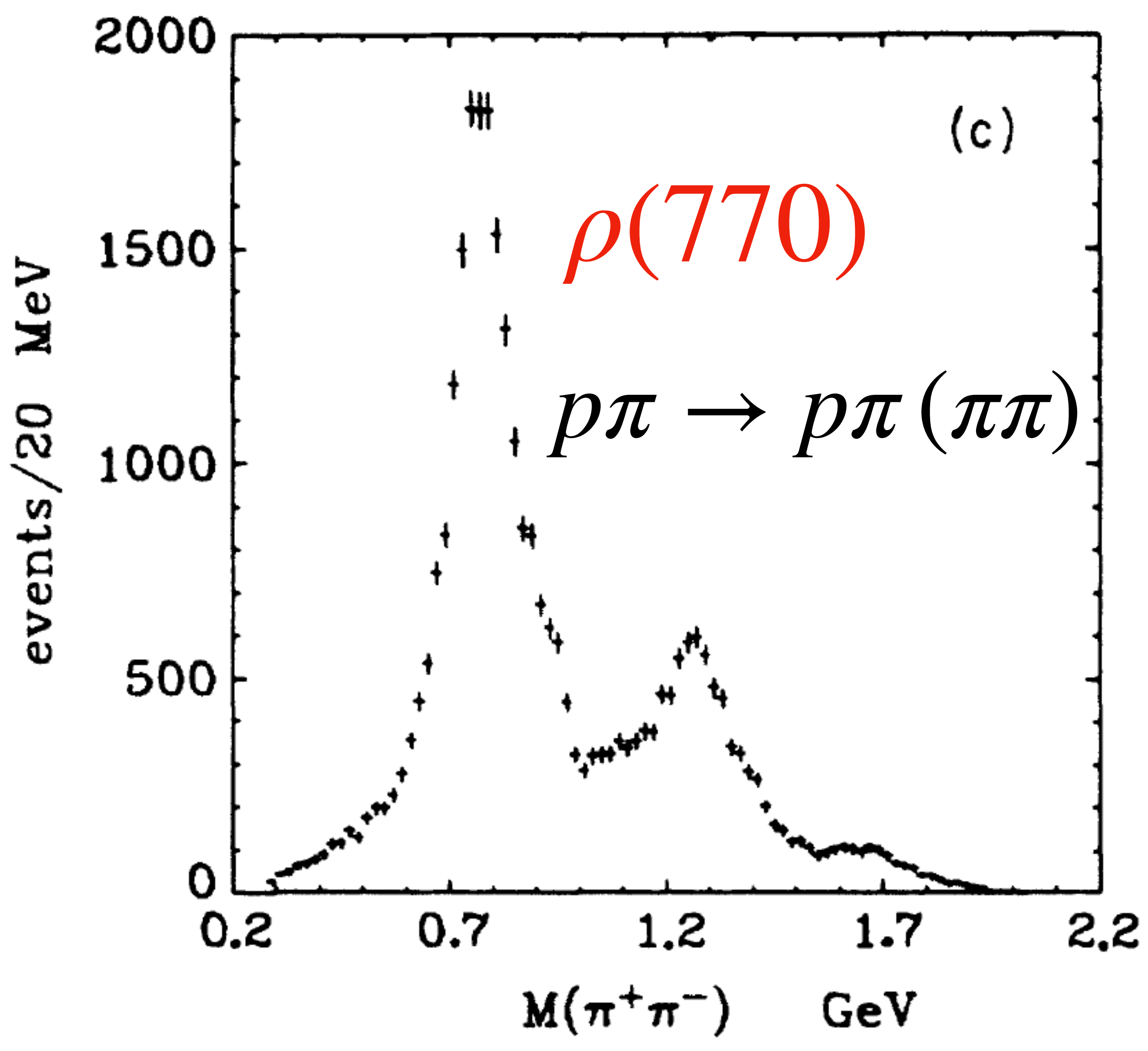

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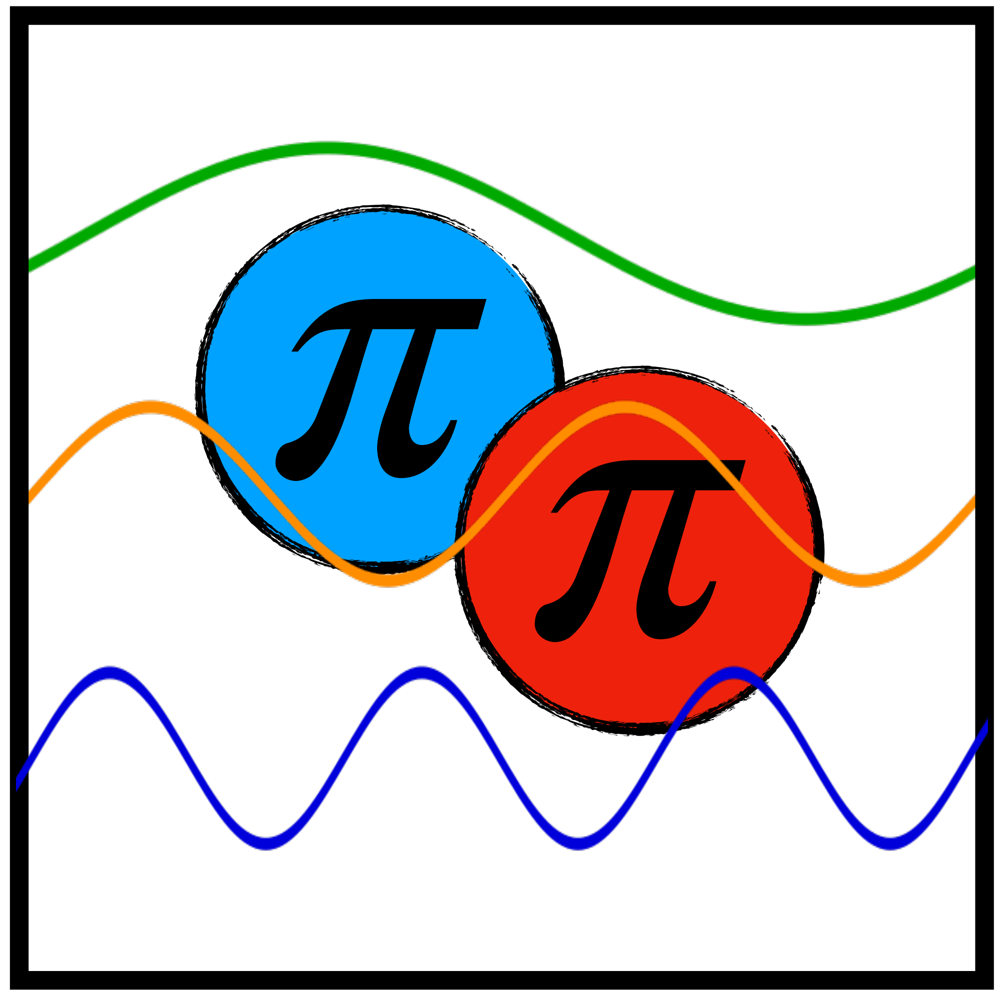
Finite-volume formalism ←

[Lüscher, 89’]



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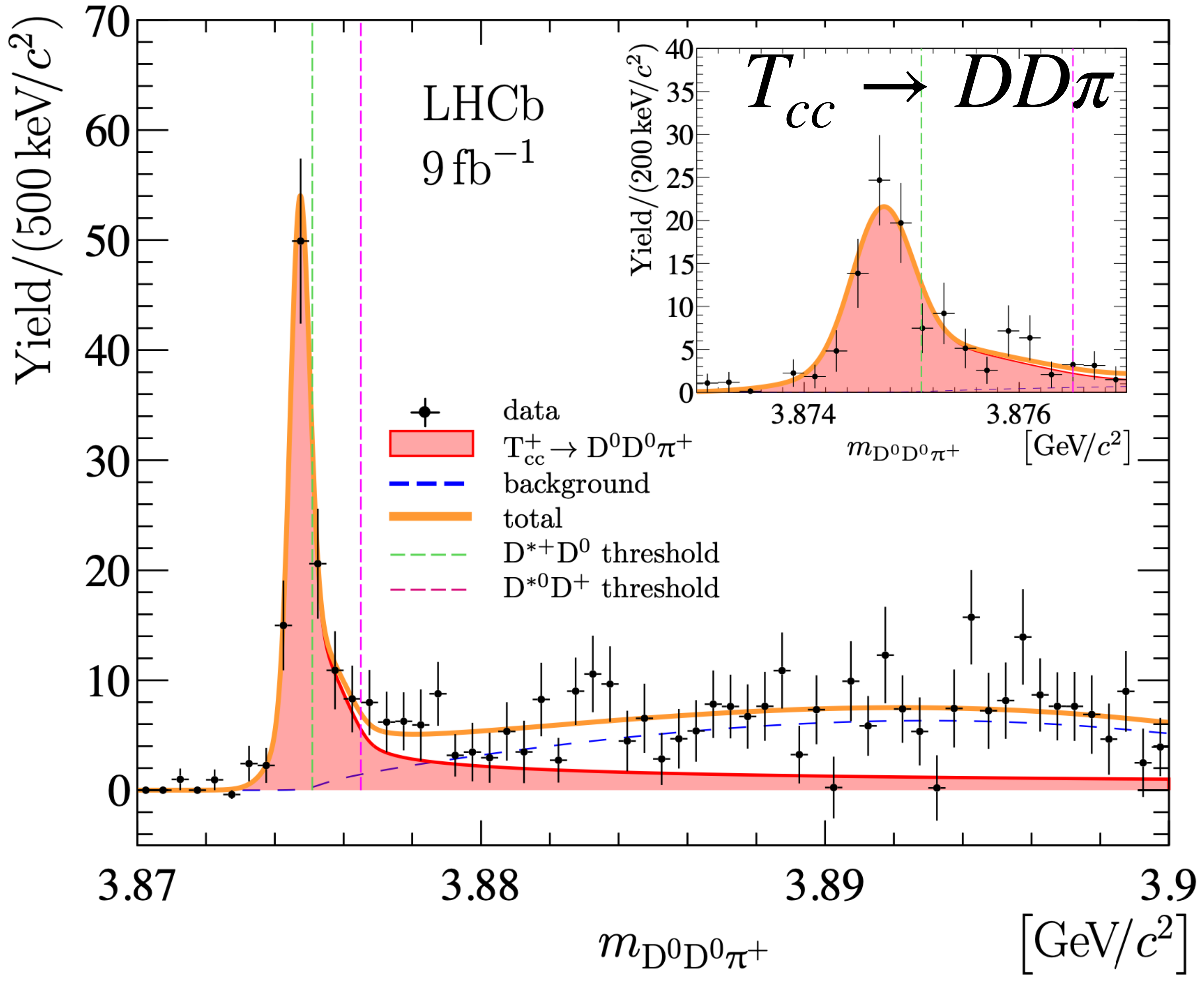
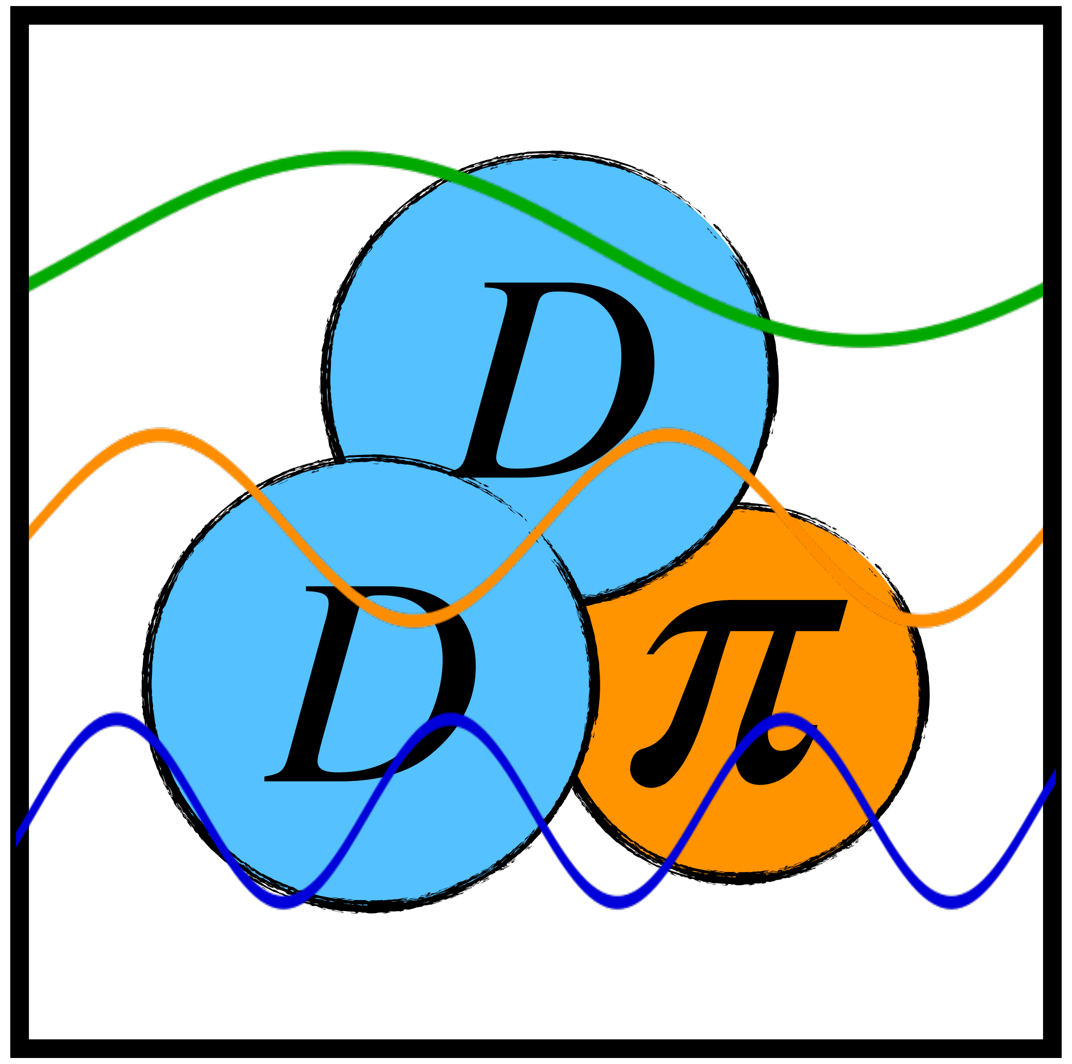


Finite-volume formalism

←

Need to include 3-body effects!



Towards the QCD S-Matrix

The S-Matrix contains the physical information of the theory:

$$S_{ab}(E) \equiv \langle \mathrm{out}|\mathcal{S}|\mathrm{in}\rangle$$

Lattice QCD → QCD S-matrix

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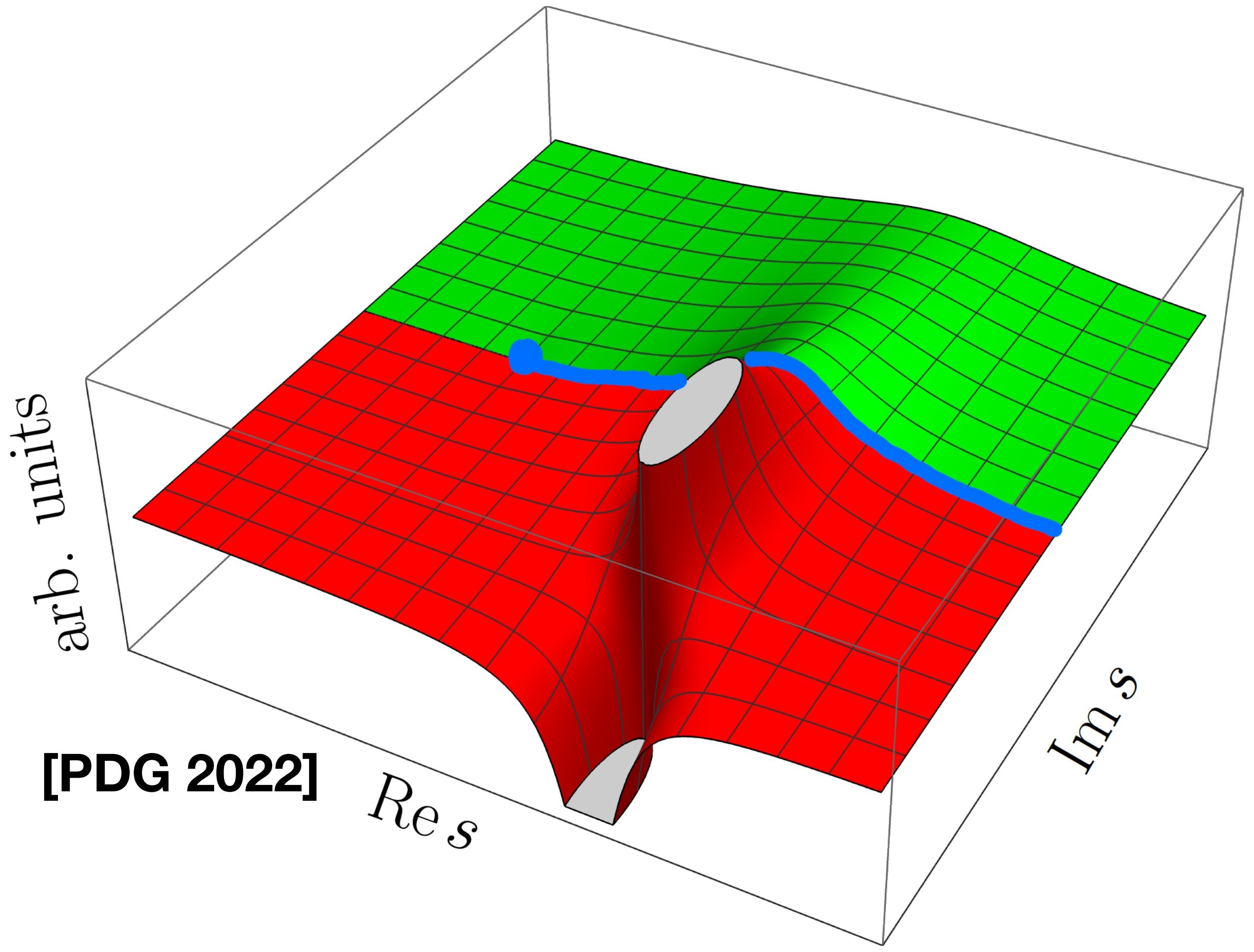
Lattice QCD ⟶ QCD S-matrix

- Resonances as poles in the S-matrix (or scattering amplitude)



[PDG 2022]

$$\sim \frac{g}{E^2 - E_R^2}$$

$$E_R = M_R - i\,\Gamma/2$$

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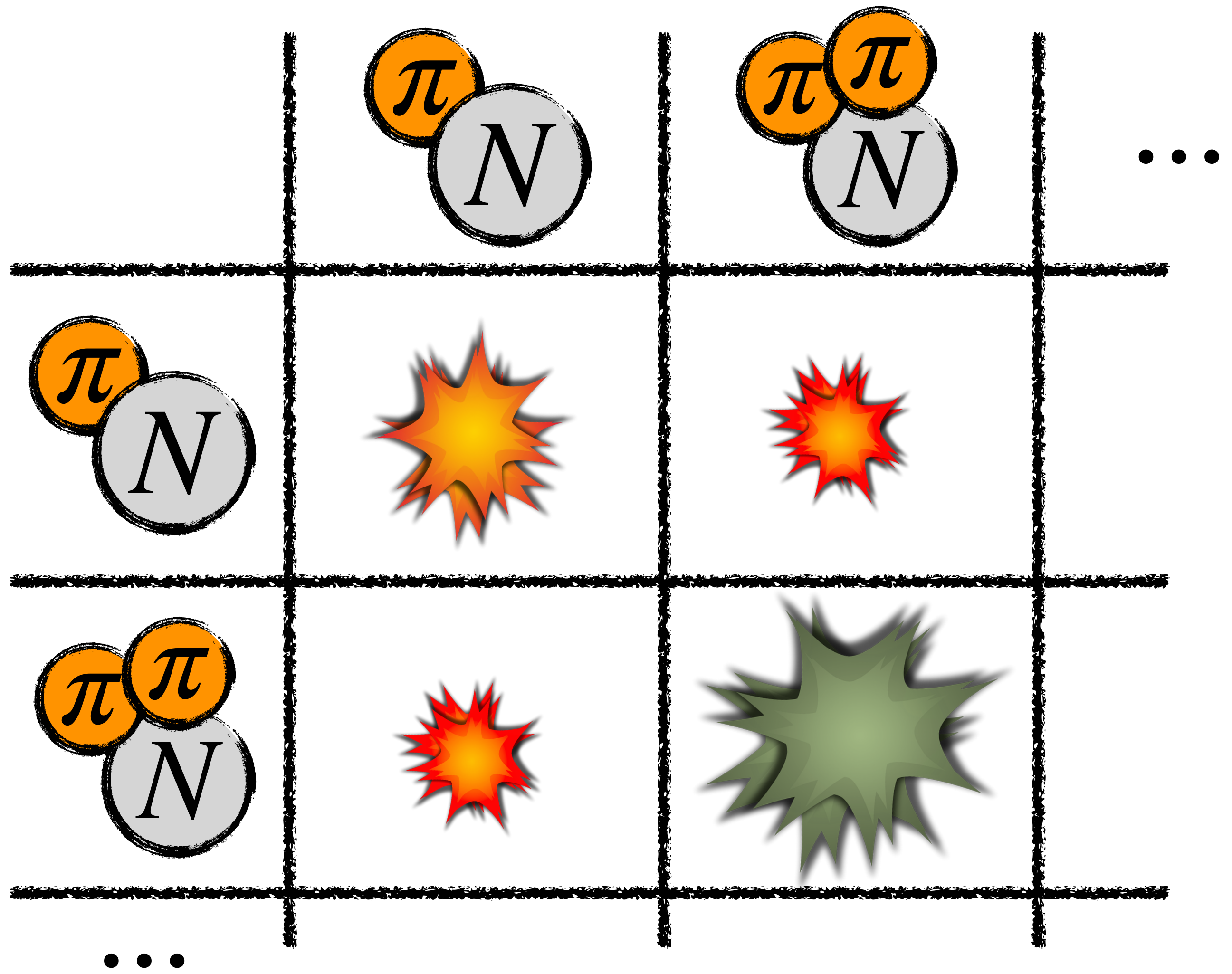
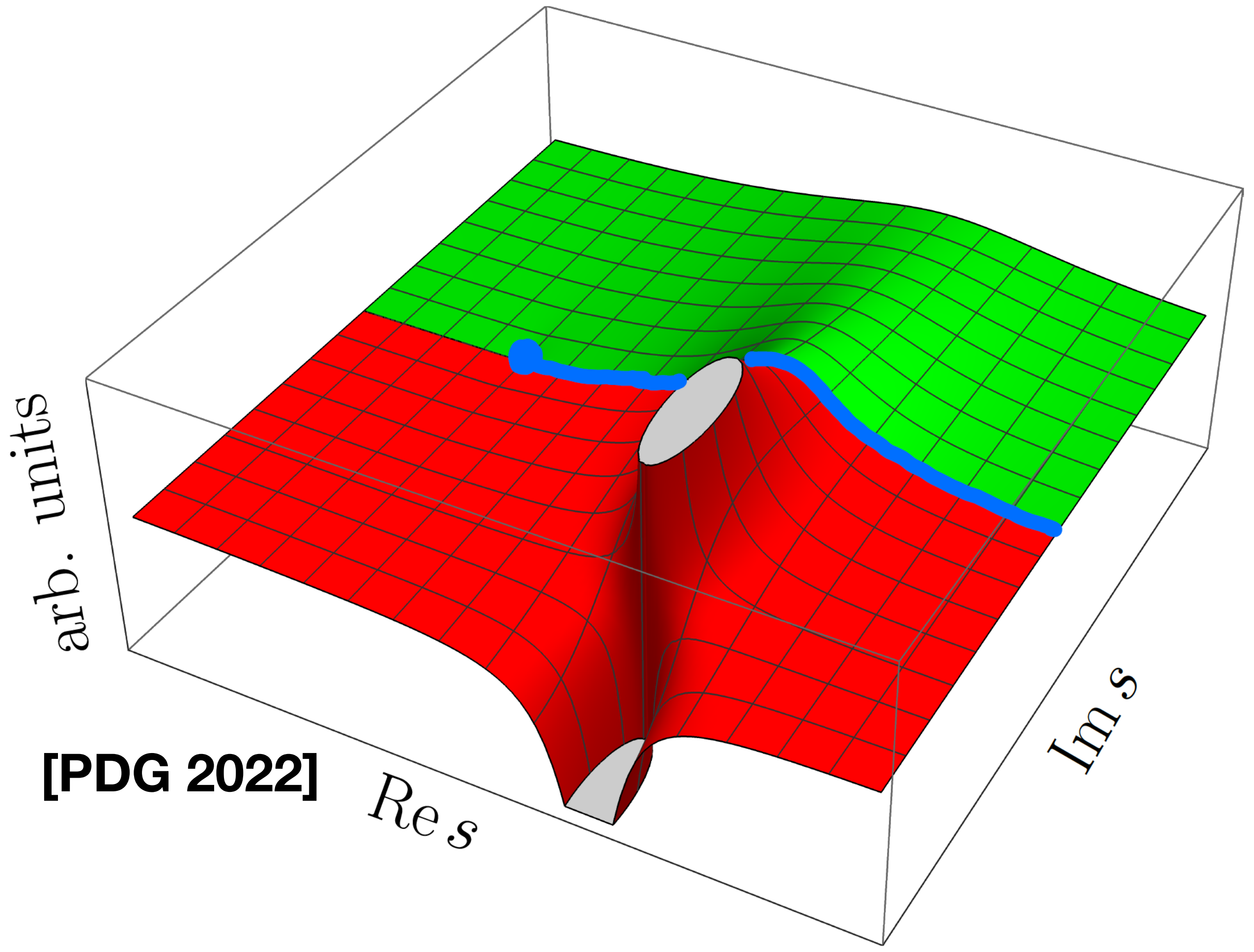


- **Resonances as poles in the S-matrix (or scattering amplitude)**



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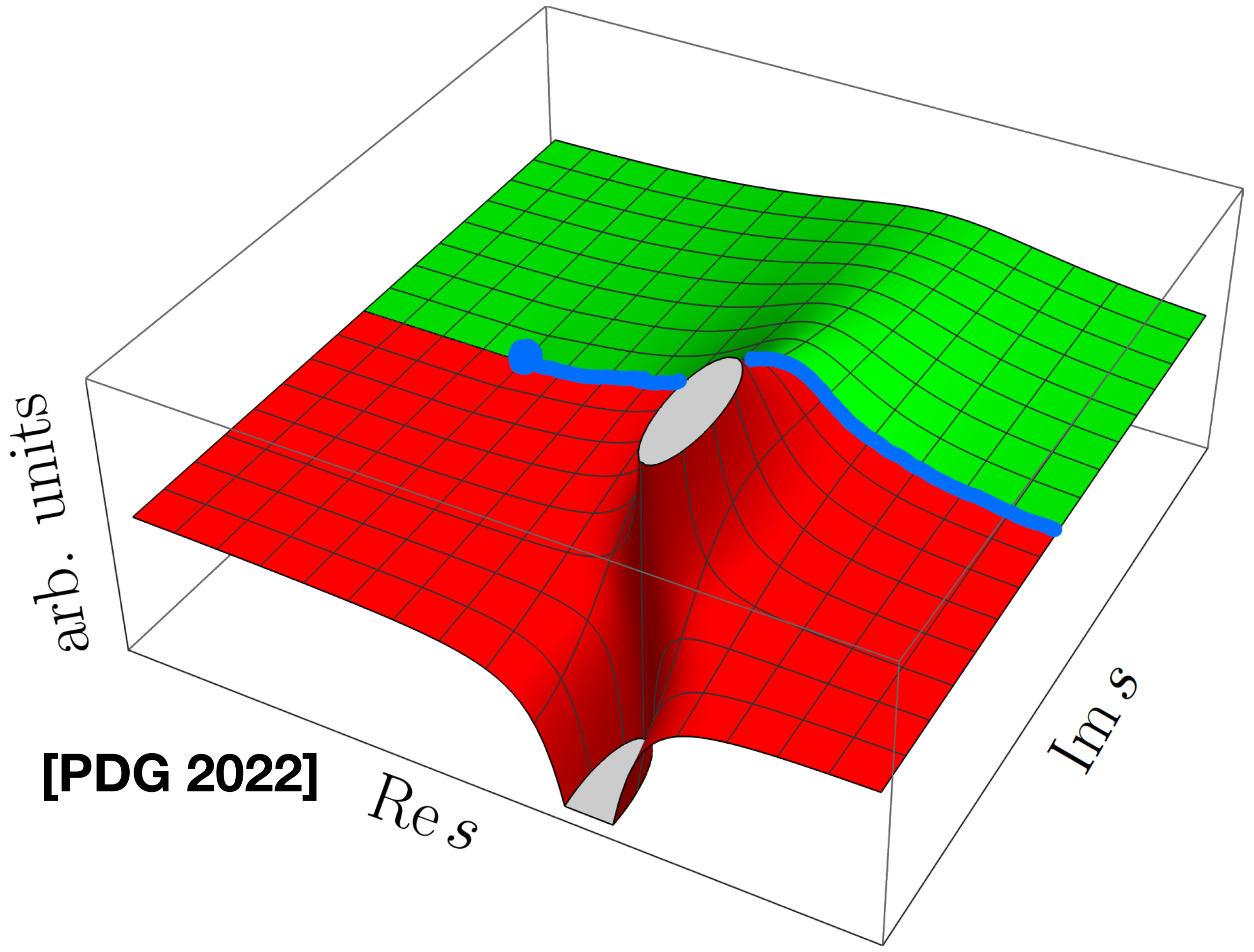

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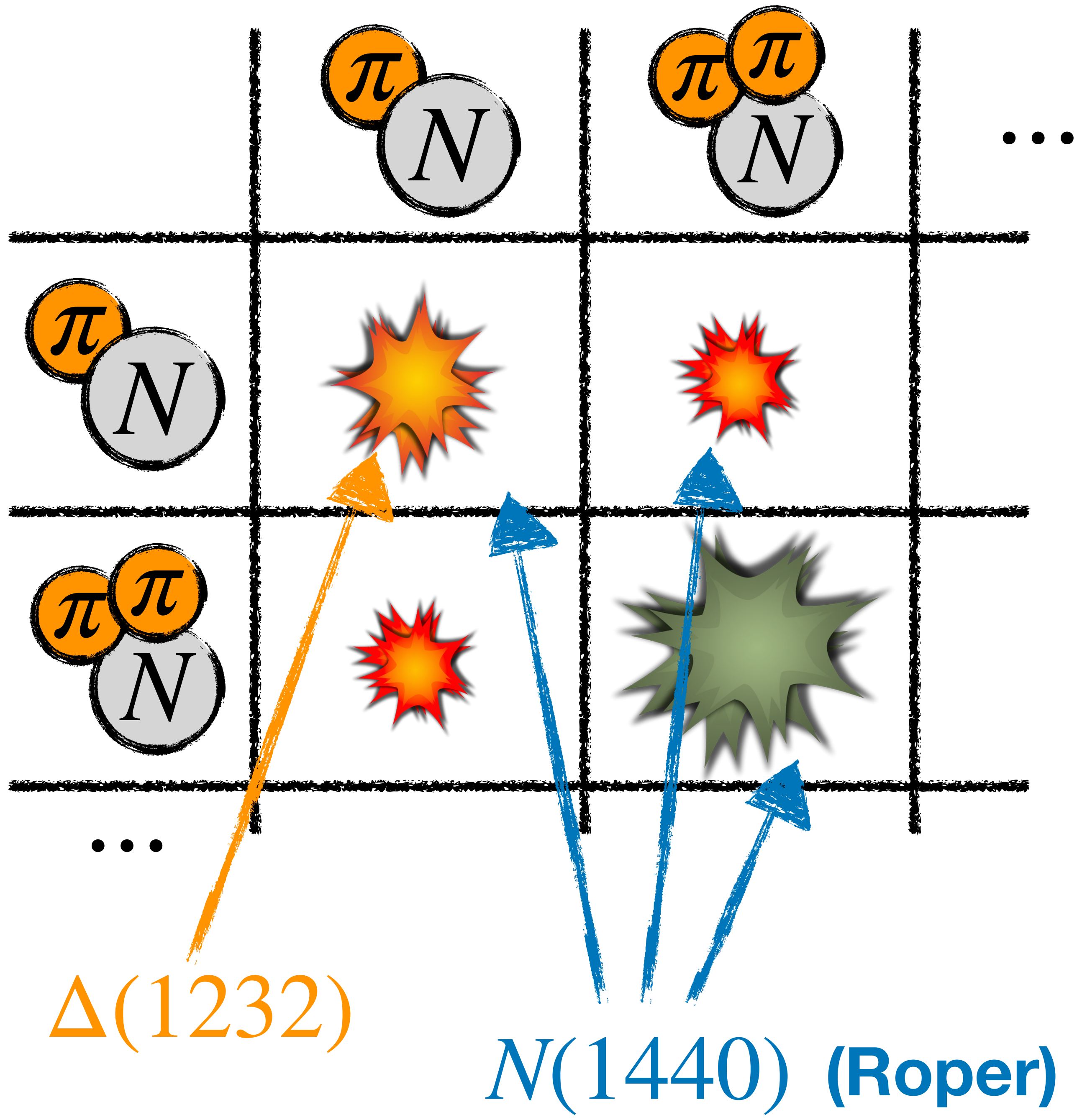


$\Delta(1232)$

$N(1440)$ **(Roper)**

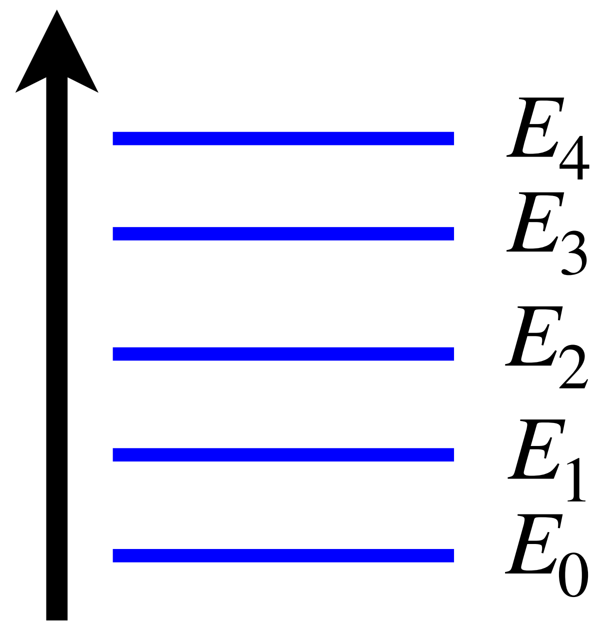
Outline

1. Three-hadron scattering from lattice QCD
2. Three mesons at maximal isospin from lattice QCD
3. A three-body description of the T_{cc}

Scattering amplitudes from lattice QCD

Formalism

Spectrum



From Euclidean

Correlation functions

Quantization condition

$$\det_{\ell m}\left[F^{-1}+\mathcal{K}\right]=0$$

A bridge between:

finite volume → infinite volume

Infinite-volume scattering quantities

K matrix	Phase shift	Amplitude
$\mathcal{K}$	δ_ℓ	$\mathcal{M}$

Resonance poles

$$\mathcal{M}^{-1}(E_{\text{pole}})=0$$

Finite-Volume Spectrum

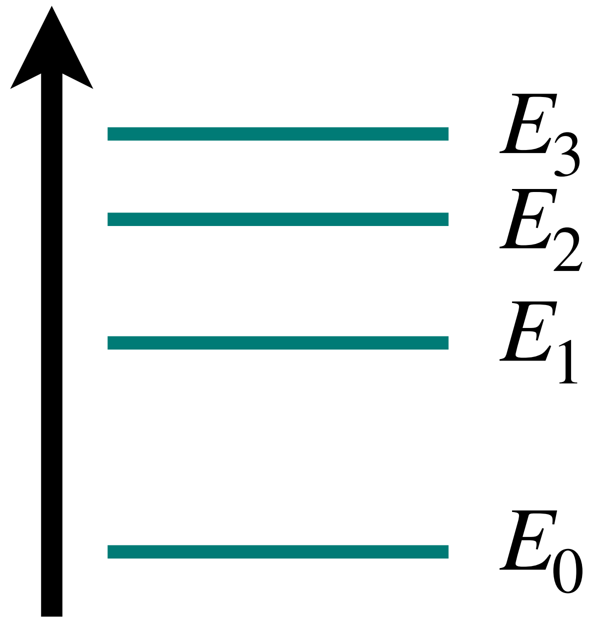
- Compute Euclidean correlation functions:

$$C_{ij}(t) = \langle \mathcal{O}_i(t)\mathcal{O}_j^\dagger(0)\rangle$$

$$C_{ij}(t) = \langle 0|\mathcal{O}_i|n\rangle\langle n|\mathcal{O}_j|0\rangle^* e^{-E_n t}$$

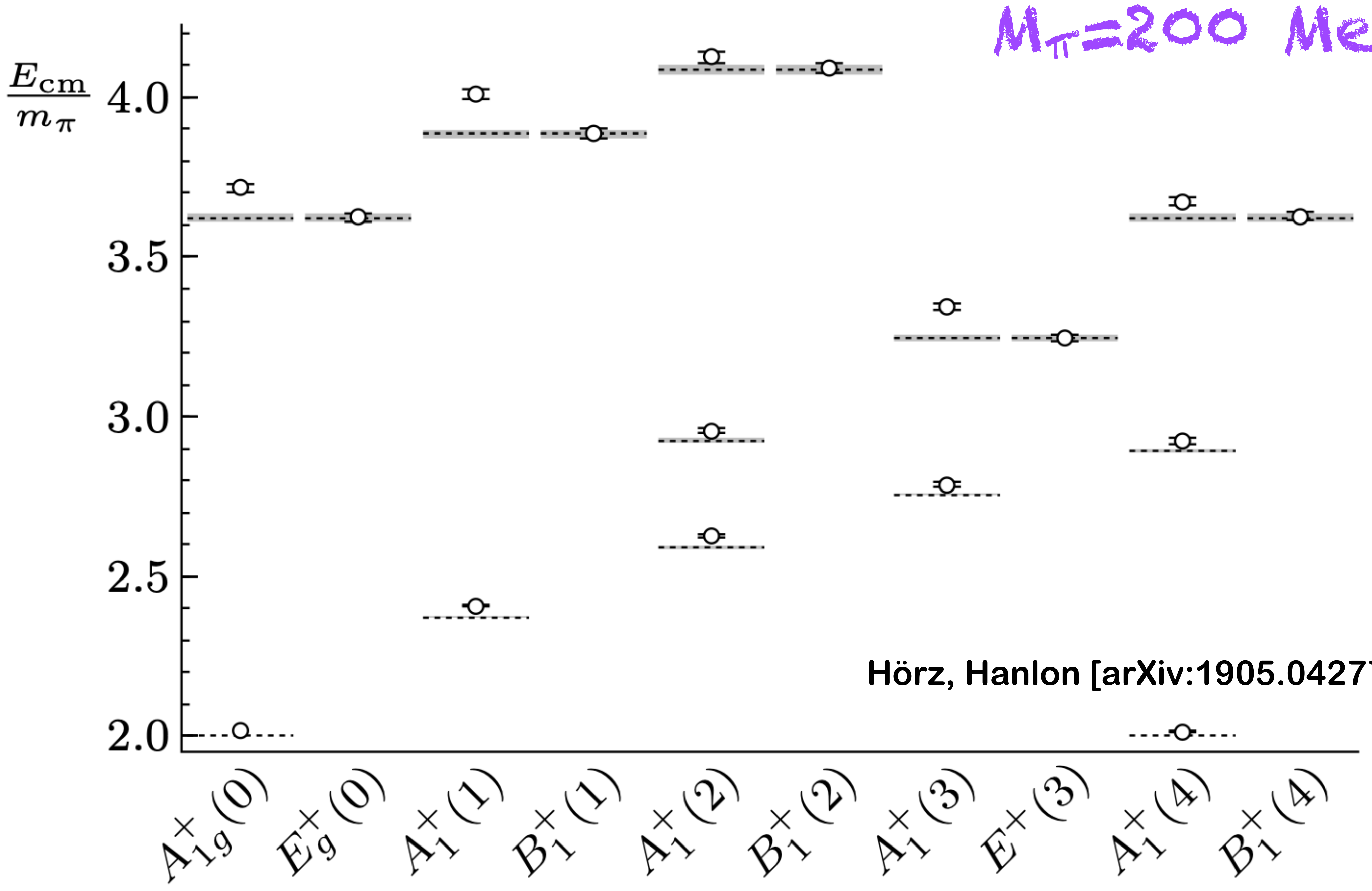
- Variational techniques
  (Generalized EigenValue Problem, GEVP)

The Spectrum



$\mathcal{O} \sim \pi^+\pi^+$

M_π=200 MeV

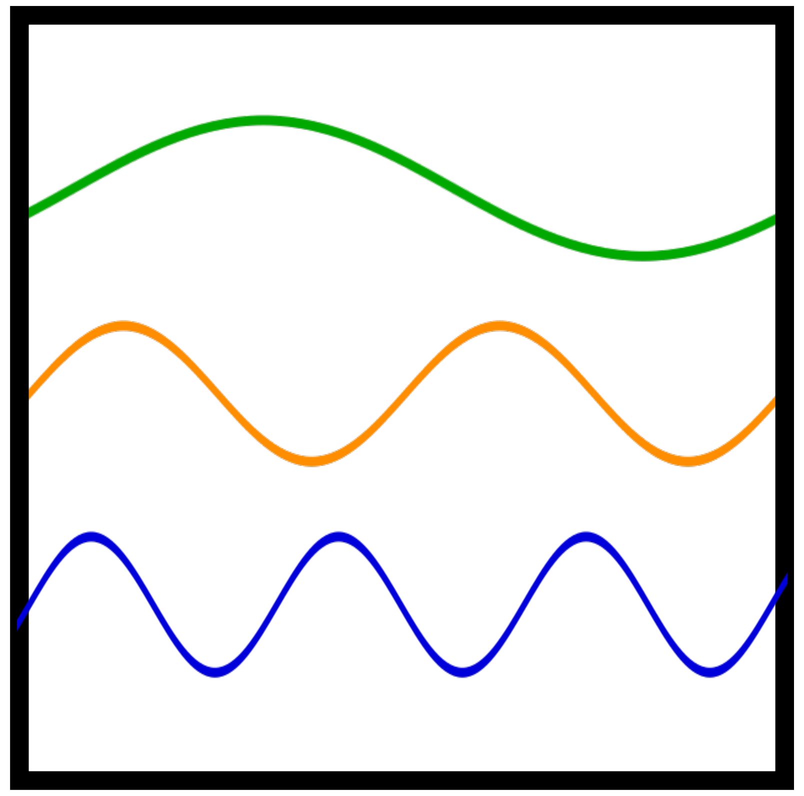


Hörz, Hanlon [arXiv:1905.04277]

Irreps = “finite-volume quantum numbers”

Finite-Volume Spectrum

Free scalar particles in finite volume with periodic boundaries



$$\vec{p} = \frac{2\pi}{L}(n_x, n_y, n_z)$$

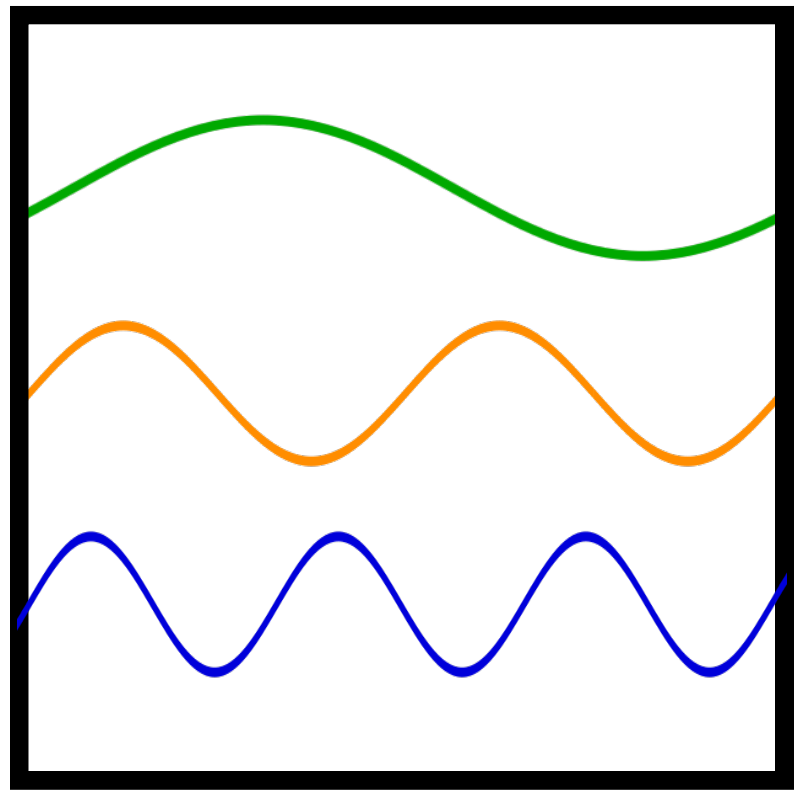
Two particles: $E = 2\sqrt{m^2 + \frac{4\pi^2}{L^2}\vec{n}^2}$

Finite-Volume Spectrum

Free scalar particles in finite volume
with periodic boundaries

→

Interactions change the spectrum:
it can be treated as a perturbation



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Ground state to leading order

$$E_2 - 2m = \langle \phi(\vec{0})\phi(\vec{0}) \,|\, \mathbf{H_I} \,|\, \phi(\vec{0})\phi(\vec{0}) \rangle$$

$$\Delta E_2 = \frac{\mathcal{M}_2(E = 2m)}{8m^2L^3} + O(L^{-4})$$

[Huang, Yang, 1958]

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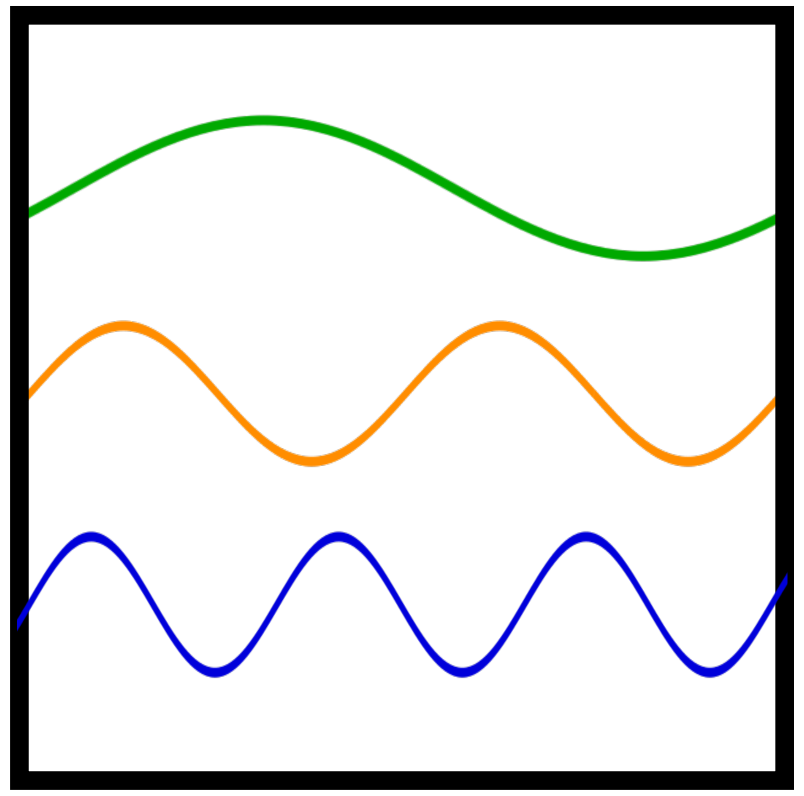

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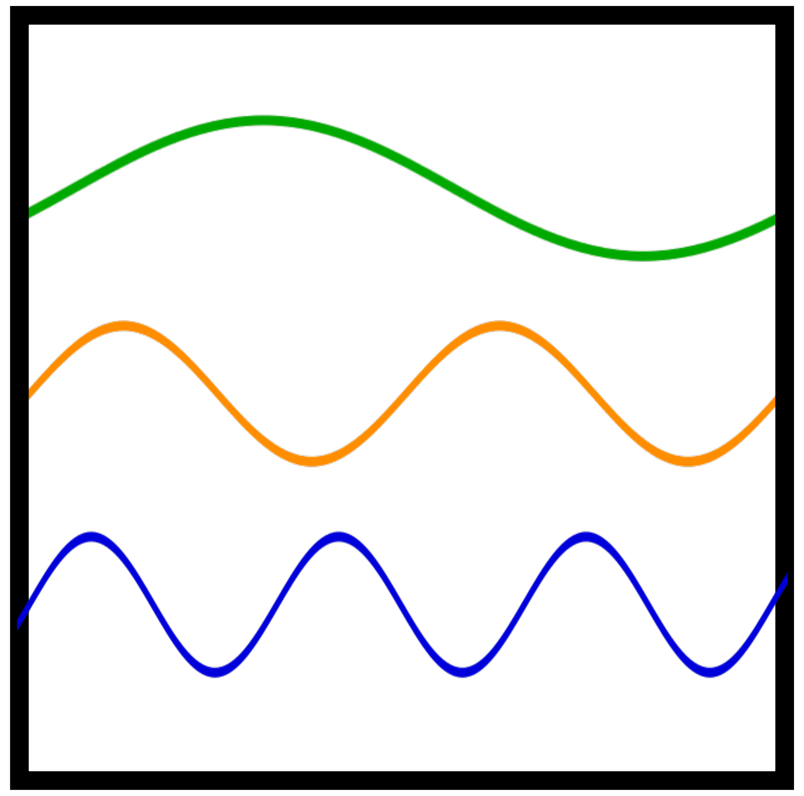
The energy shift of the two-particle ground state is related to the $2 \to 2$ scattering amplitude

Finite-Volume Spectrum

Free scalar particles in finite volume with periodic boundaries



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In general a problem of Quantum Field Theory in finite volume

Interactions change the spectrum: it can be treated as a perturbation

d state to leading order

$$2m = \langle \phi(\vec{0})\phi(\vec{0}) \,|\, \mathbf{H_I} \,|\, \phi(\vec{0})\phi(\vec{0}) \rangle$$

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[Huang, Yang, 1958]

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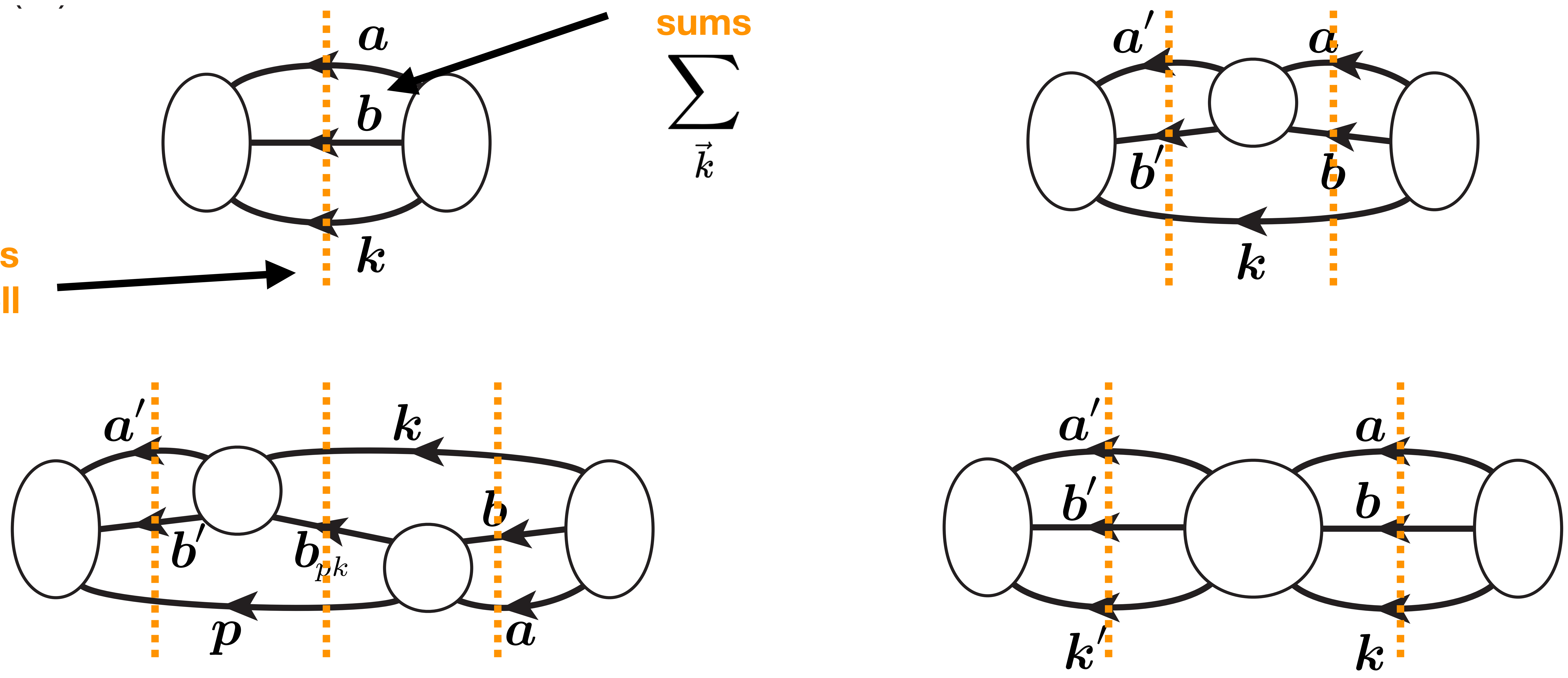
Finite-Volume Effects

- **Derivation in an EFT to all orders including finite-volume effects from all diagrams**
  [Hansen, Sharpe, PRD 2014 & 2015]



momentum sums

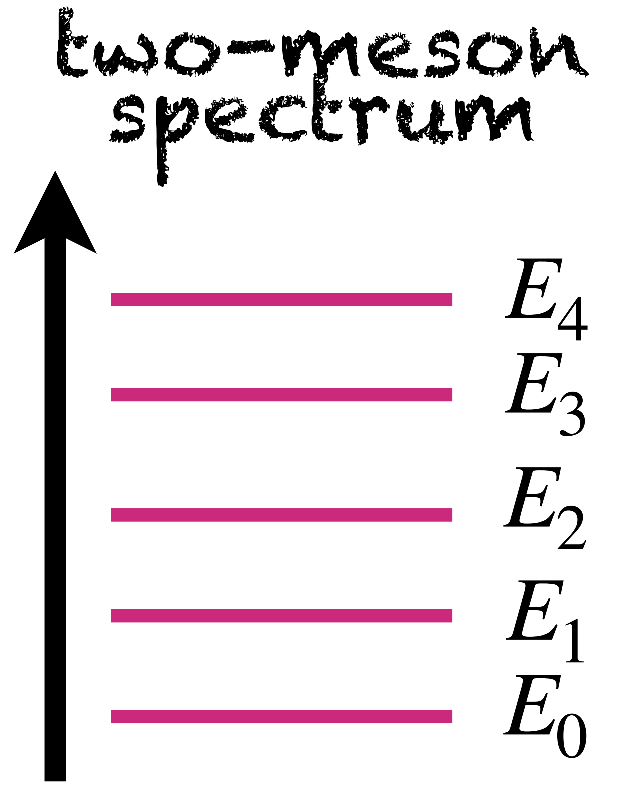
$$\sum_{\vec{k}}$$

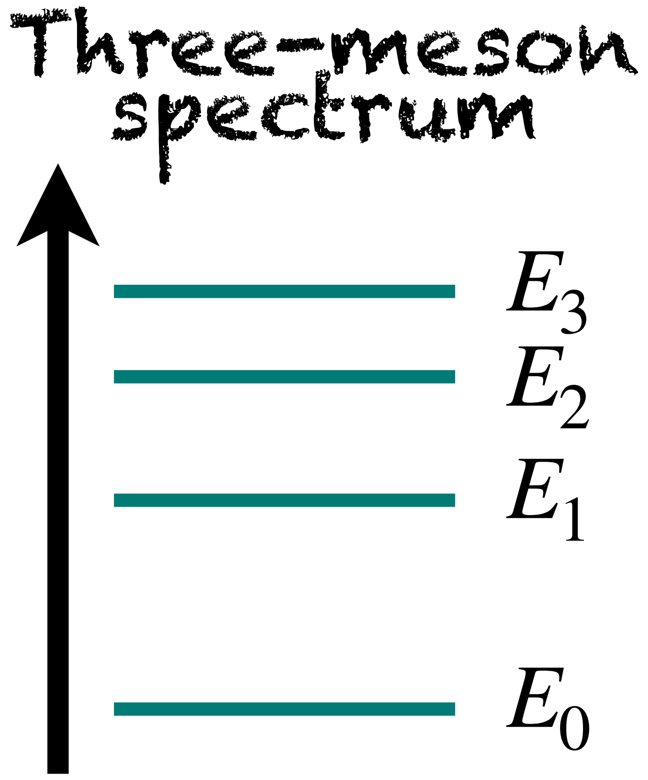
Three particles can be on shell

[Draper, Hansen, FRL, Sharpe, JHEP 2023]

Formalism

[Hansen, Sharpe, PRD 2014 & 2015]





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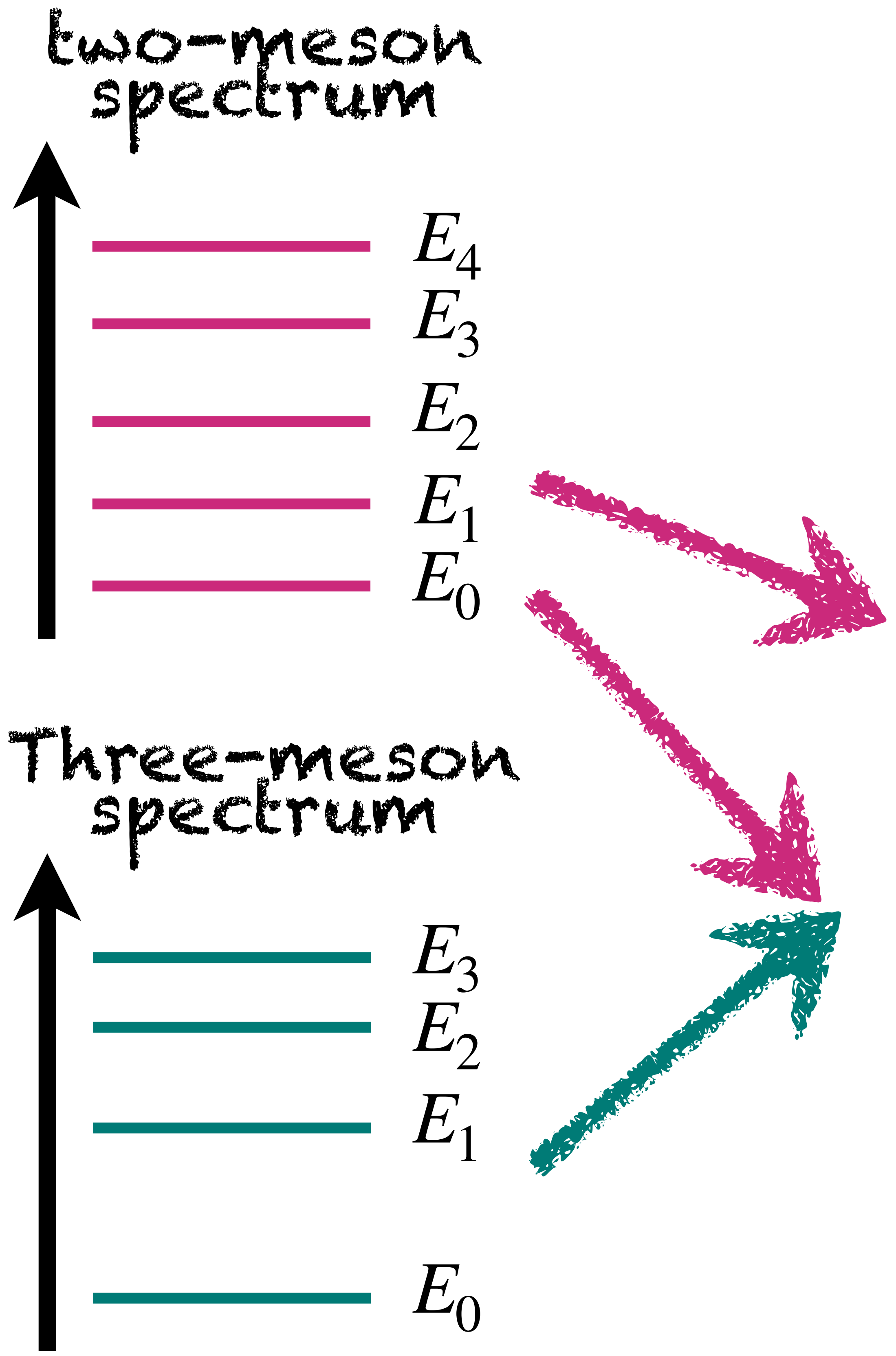


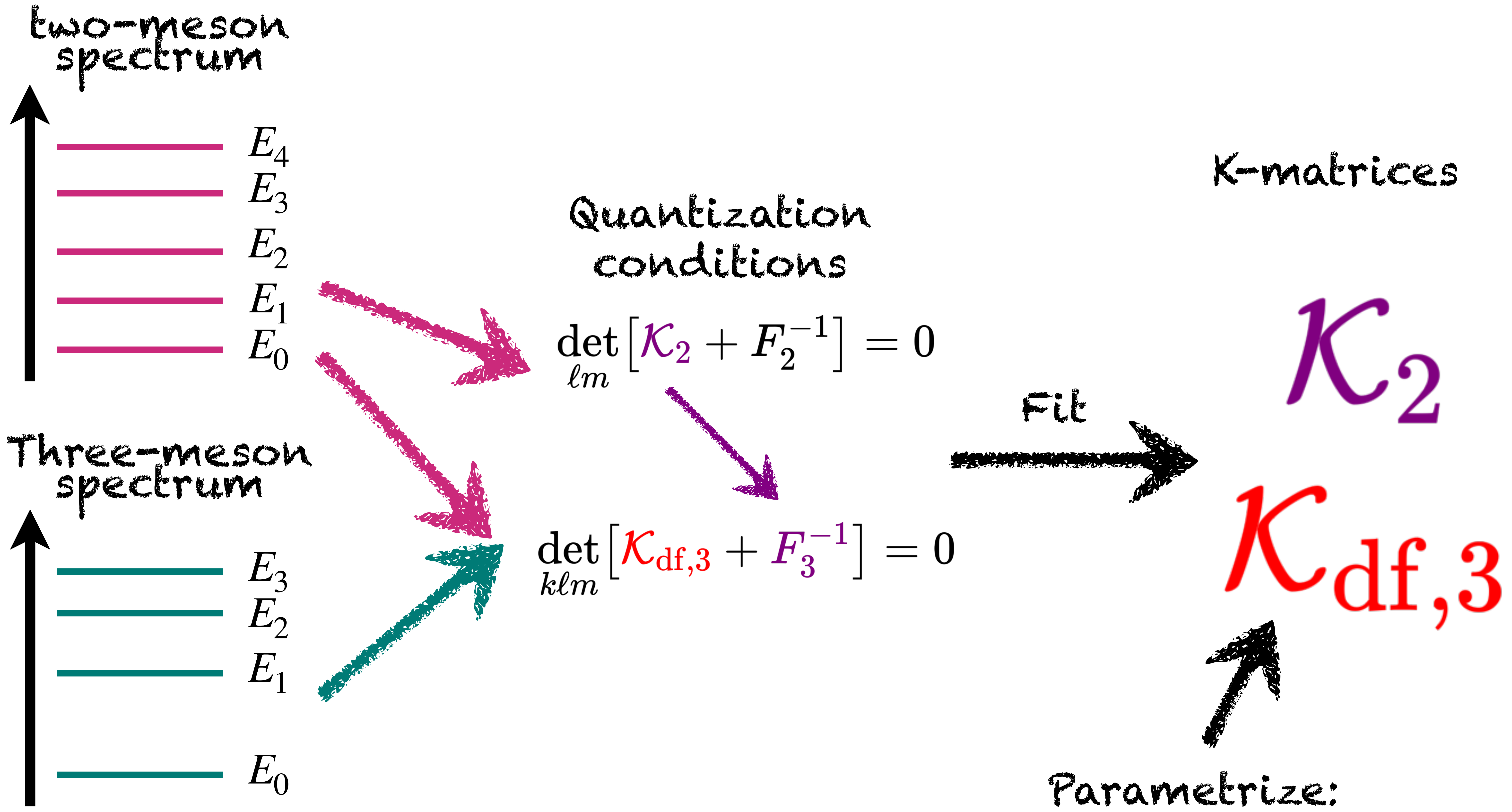
Quantization conditions

$$\det_{\ell m}\left[\mathcal{K}_2 + F_2^{-1}\right] = 0$$

$$\det_{k\ell m}\left[\mathcal{K}_{\mathrm{df},3} + F_3^{-1}\right] = 0$$

Formalism

[Hansen, Sharpe, PRD 2014 & 2015]



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K-matrices

$$\mathcal{K}_2$$

$$\mathcal{K}_{\mathrm{df},3}$$

Parametrize:

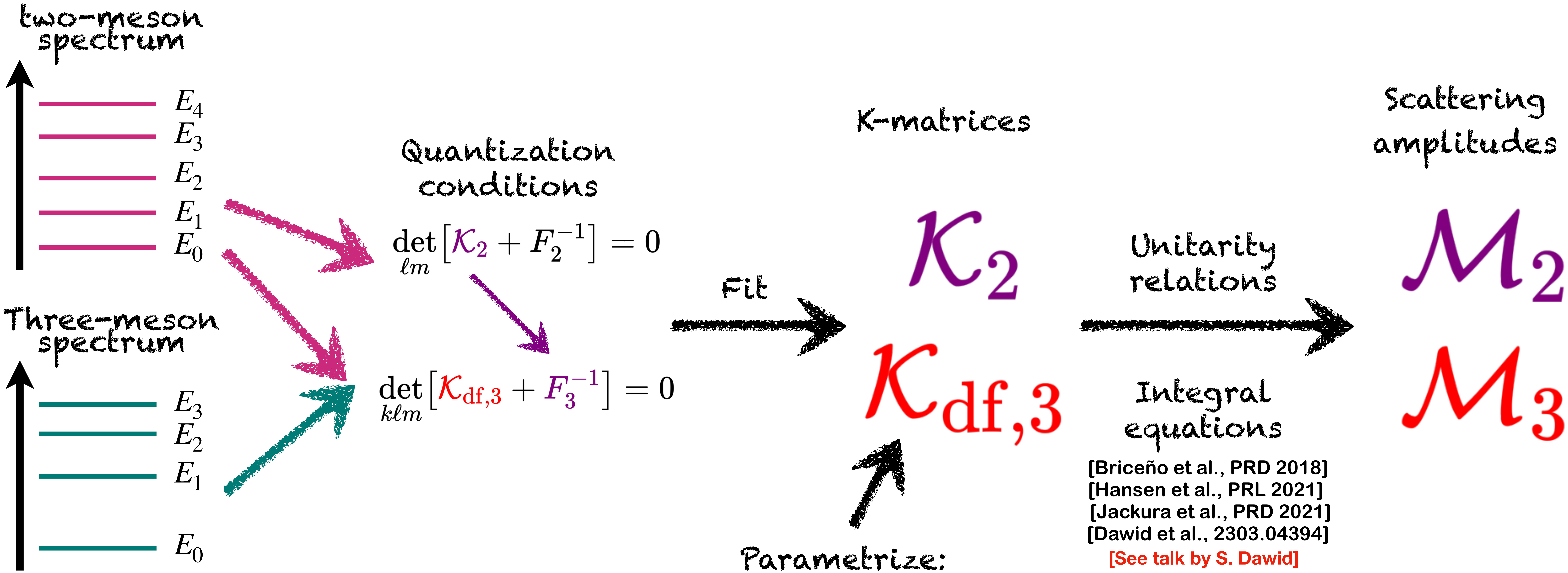
$$\mathcal{K}_2 = c_0 + c_1 k^2 + \ldots$$

$$\mathcal{K}_{\mathrm{df},3} = \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},0} + \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},1}\left(\frac{s-9m^2}{9m^2}\right) + \ldots$$

[Blanton, FRL, Sharpe, JHEP 2019]

Formalism

[Hansen, Sharpe, PRD 2014 & 2015]



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Fit

K-matrices

$$\mathcal{K}_2$$

$$\mathcal{K}_{\mathrm{df},3}$$

Unitarity relations

Integral equations

[Briceño et al., PRD 2018]
[Hansen et al., PRL 2021]
[Jackura et al., PRD 2021]
[Dawid et al., 2303.04394]
[See talk by S. Dawid]

Scattering amplitudes

$$\mathcal{M}_2$$

$$\mathcal{M}_3$$

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[Blanton, FRL, Sharpe, JHEP 2019]

Three-mesons at maximal isospin

Three-meson systems

Important benchmark system: three pseudoscalar mesons at maximal isospin

- **Implement formalism and explore its features**
- **Test fitting strategies to extract three-body K matrix**
- **Interpret results in combination with EFTs**
- **Investigate features of scattering amplitudes**
- **First determinations of these scattering amplitudes at the physical point!**

$$\boxed{3\pi^+, \quad 3K^+, \quad \pi^+\pi^+K^+, \quad K^+K^+\pi^+}$$

[Blanton ... FRL ... et al., PRL 2020 & JHEP 2021]
[Draper ... FRL ... et al., JHEP 2023],
[Fischer ... FRL ... et al (ETMC), EPJC 2021]
[Alexandru et al, Brett et al, Culver et al, Mai et al. (GWQCD) PRD 2020&2021]
[Hansen et al (HadSpec), PRL 2021 See talk by M. Hansen]

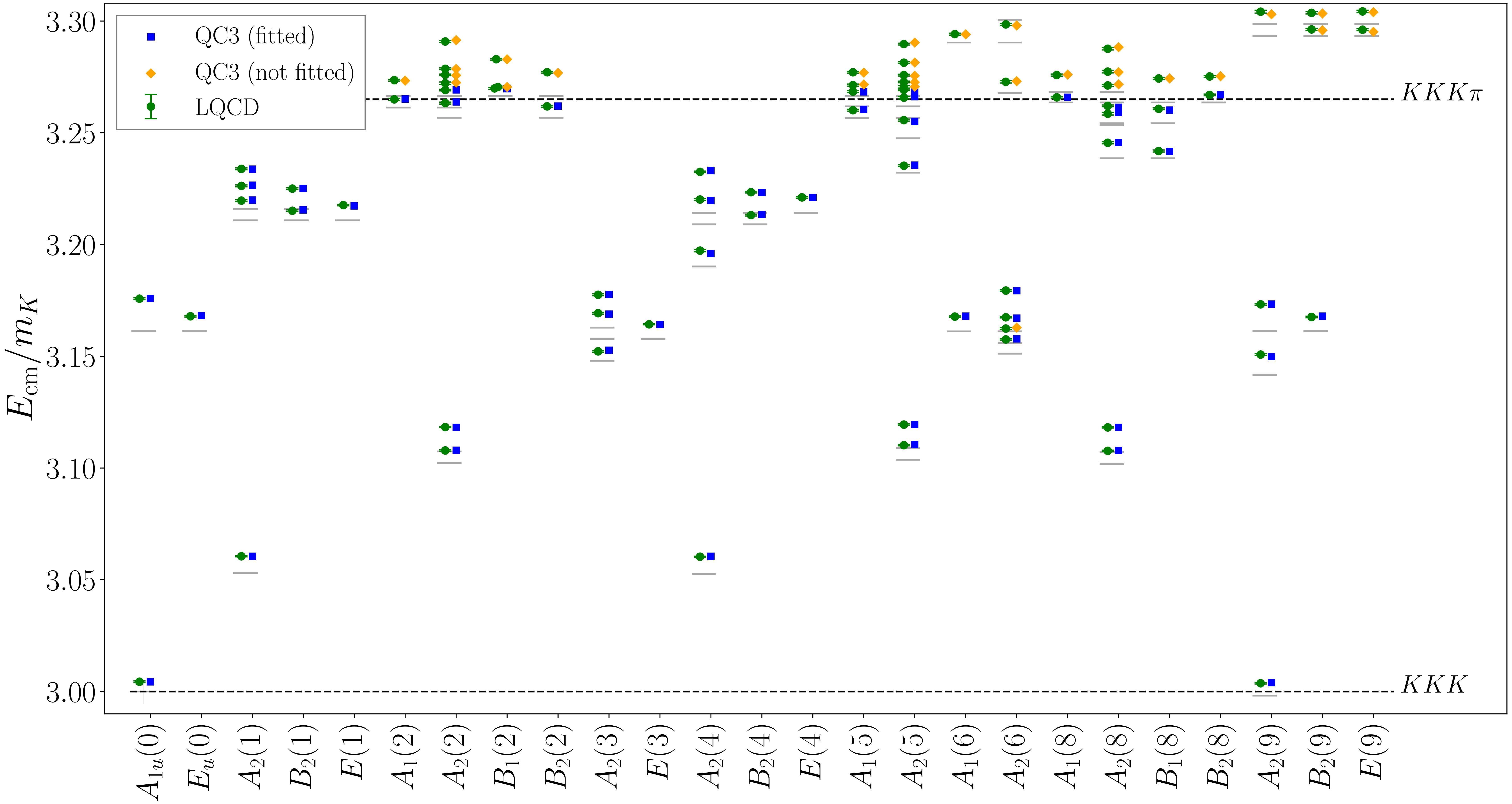
	$(L/a)^3 \times (T/a)$	M_π[MeV]	M_K[MeV]	N_{cfg}
N203	$48^3 \times 128$	340	440	771
N200	$48^3 \times 128$	280	460	1712
D200	$64^3 \times 128$	200	480	2000
E250	$96^3 \times 192$	130	500	505

$$\text{tr}\, m_q = 2m_{ud} + m_s \simeq \text{const}$$

$$a \simeq 0.063 \text{ fm}$$

3K spectrum

E250 ensemble, $M_\pi \sim 130$ MeV, $M_K \sim 500$ MeV



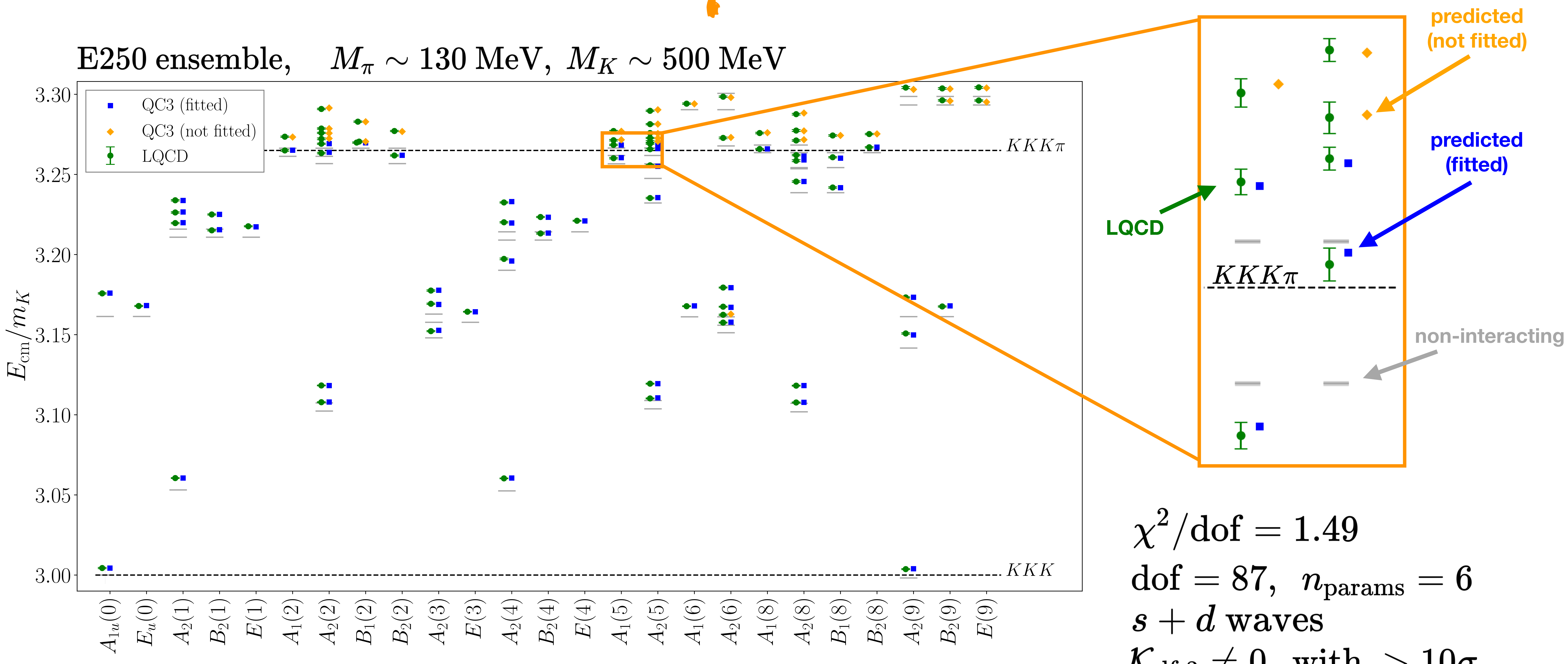
$\chi^2/\text{dof} = 1.49$

$\text{dof} = 87, \;\; n_{\text{params}} = 6$

$s + d$ waves

$\mathcal{K}_{\text{df},3} \neq 0$, with $> 10\sigma$

3K spectrum



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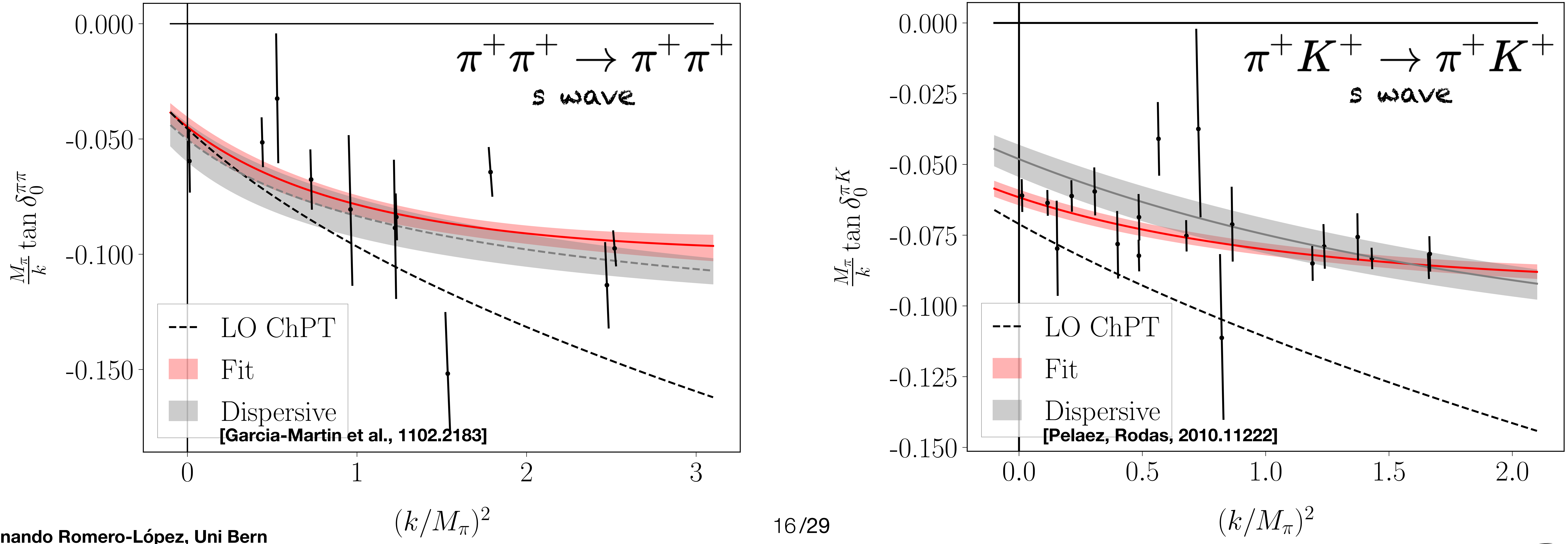
$s + d$ waves

$\mathcal{K}_{\text{df},3} \neq 0, \ \text{with} \ > 10\sigma$

Two-body phase shift

- Required input for three-meson calculations
- Competitive statistical uncertainties! ▶ Lattice QCD can access low-energy region!

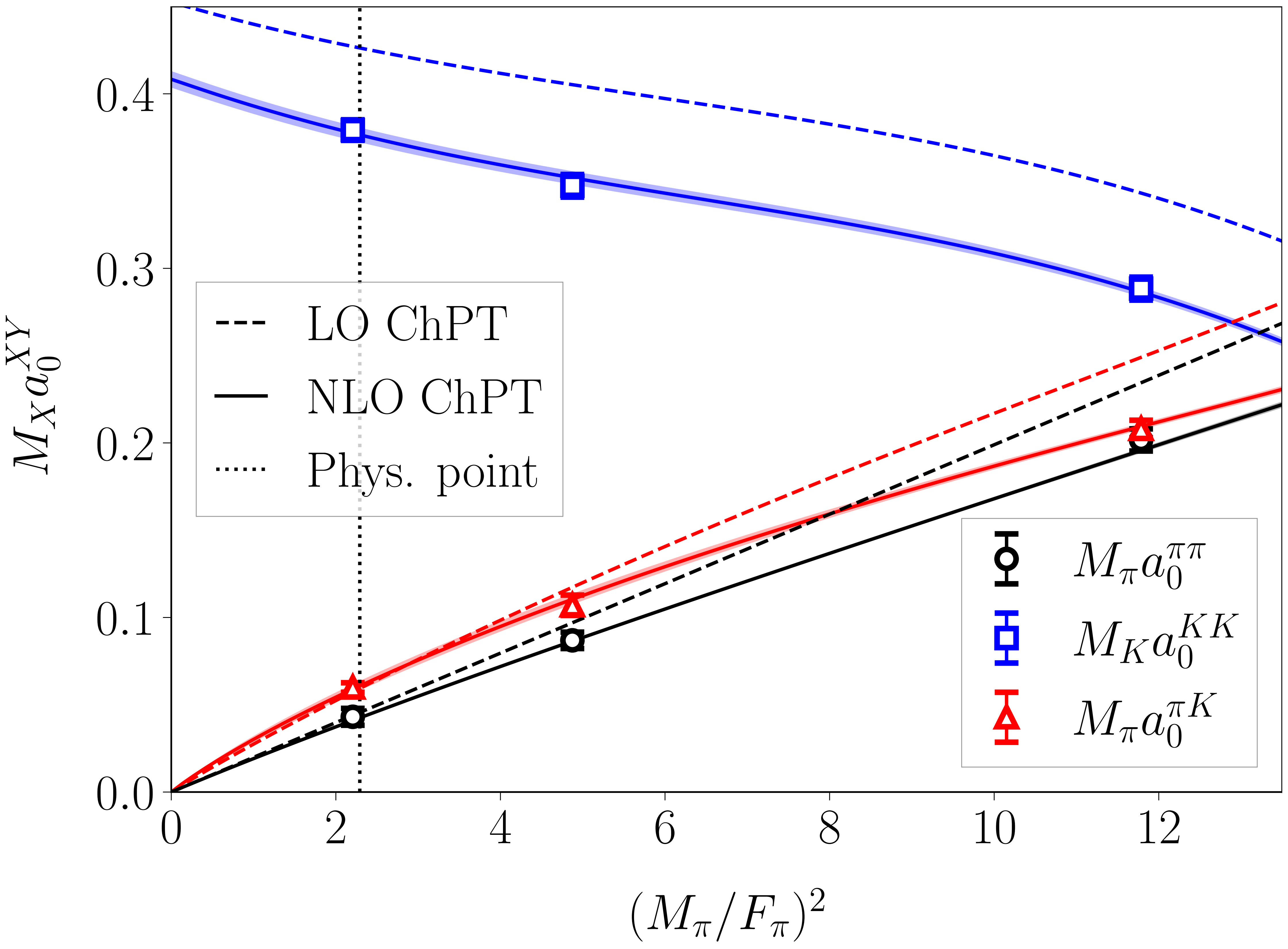


[Garcia-Martin et al., 1102.2183]

[Pelaez, Rodas, 2010.11222]

Scattering lengths

s-wave scattering lengths



Scattering lengths

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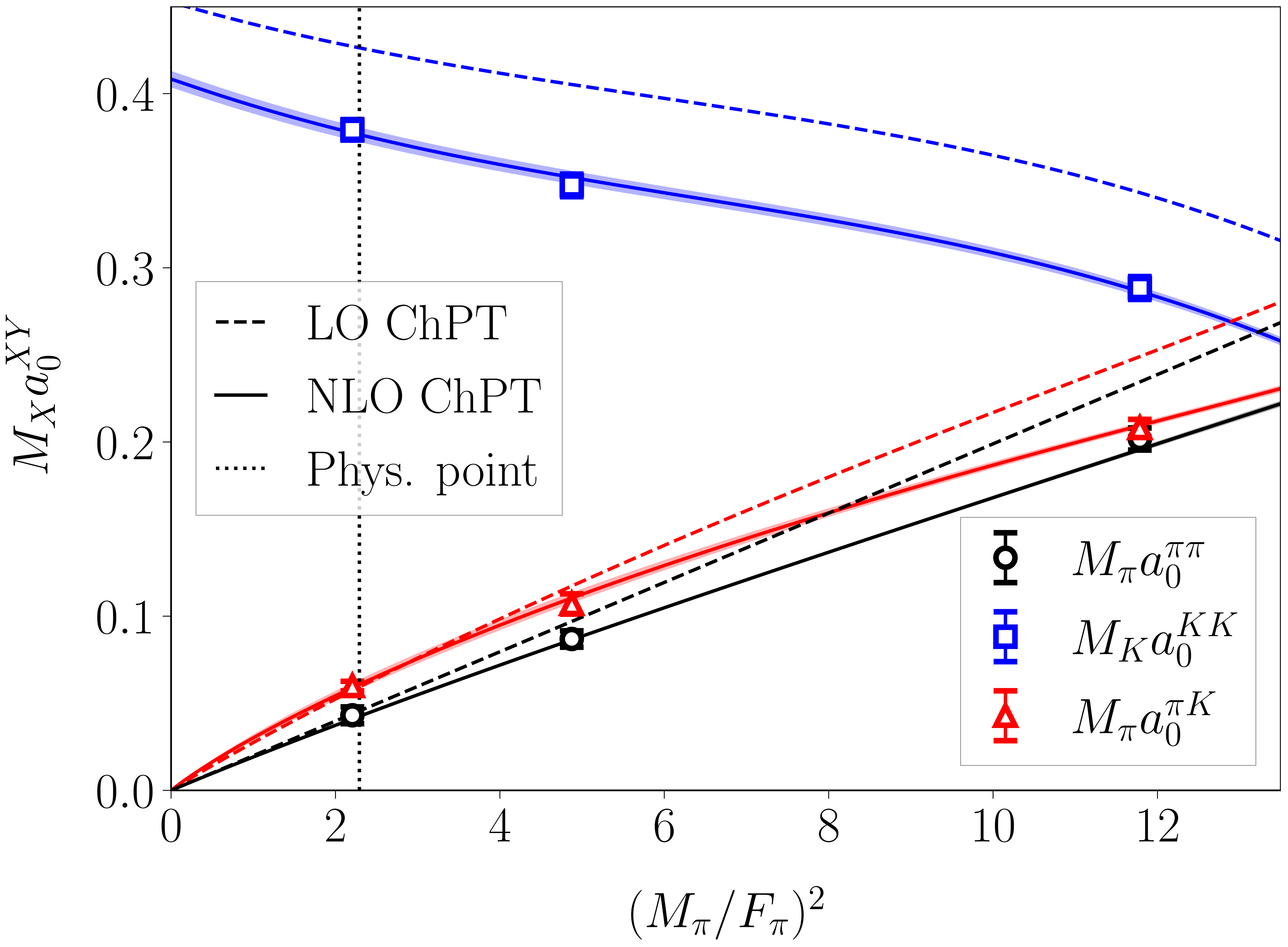
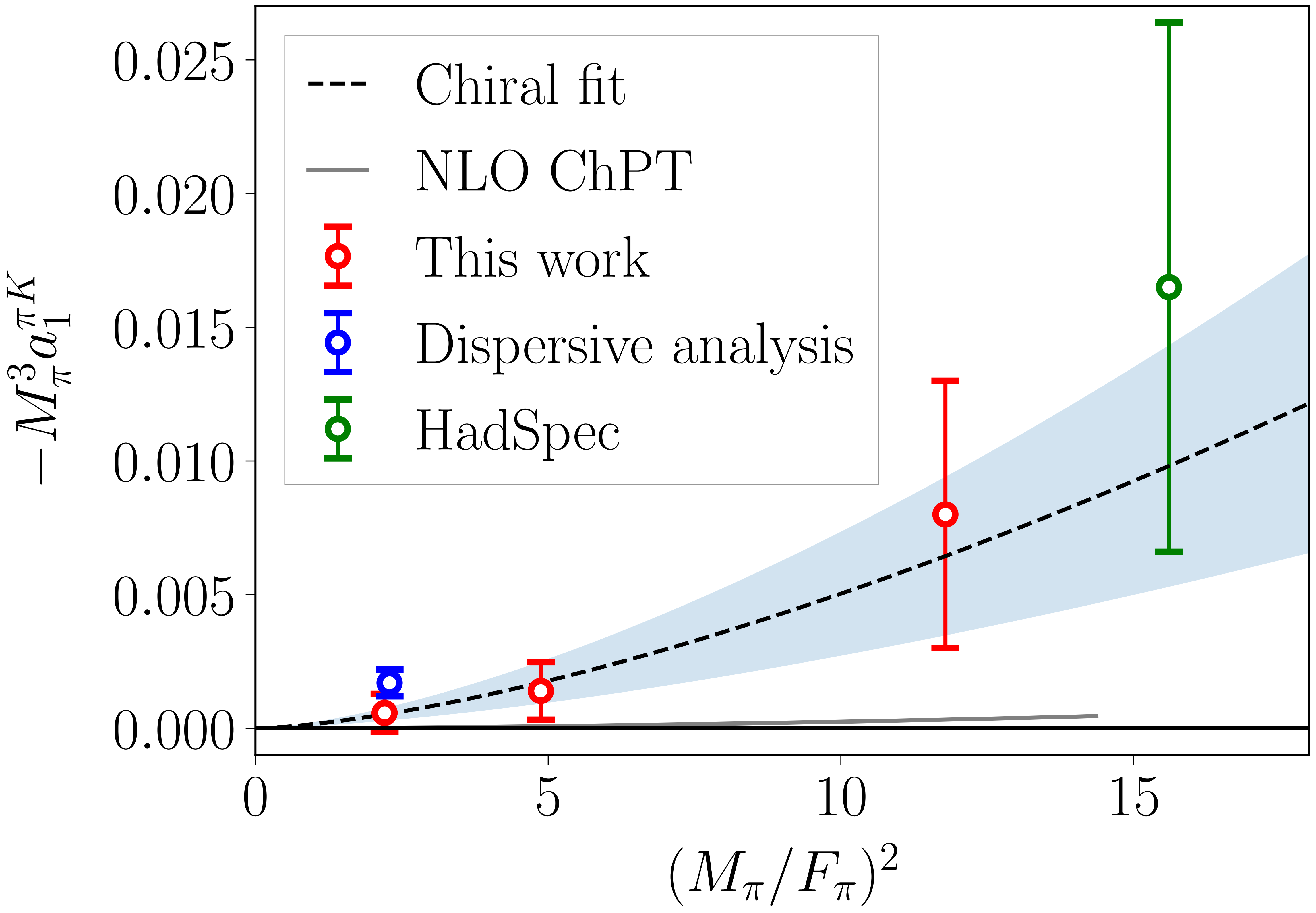


p-wave πK scattering lengths



Dispersive results: [Pelaez, Rodas, 2010.11222]

Three-pion K matrix

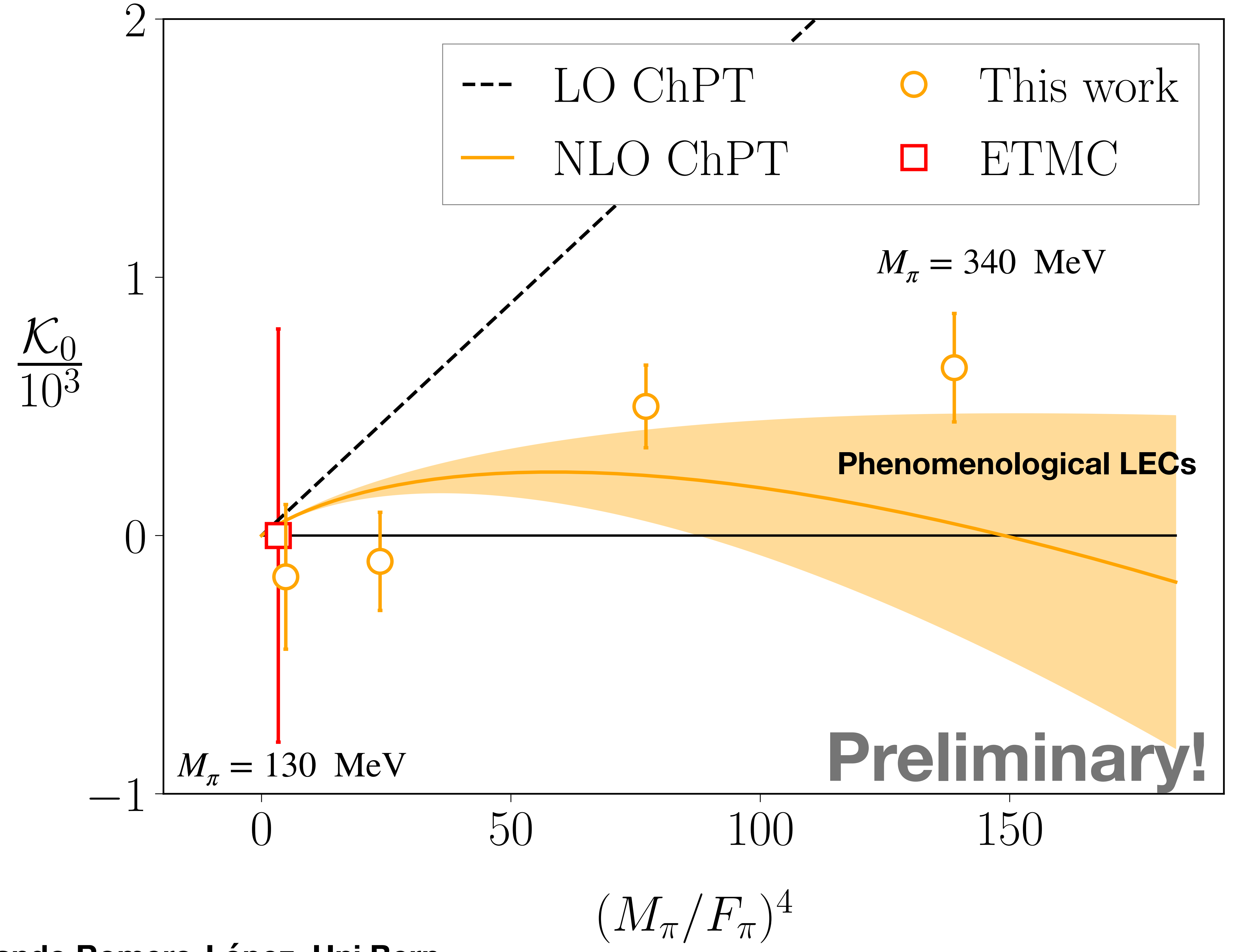
- **Compare to chiral perturbation theory**

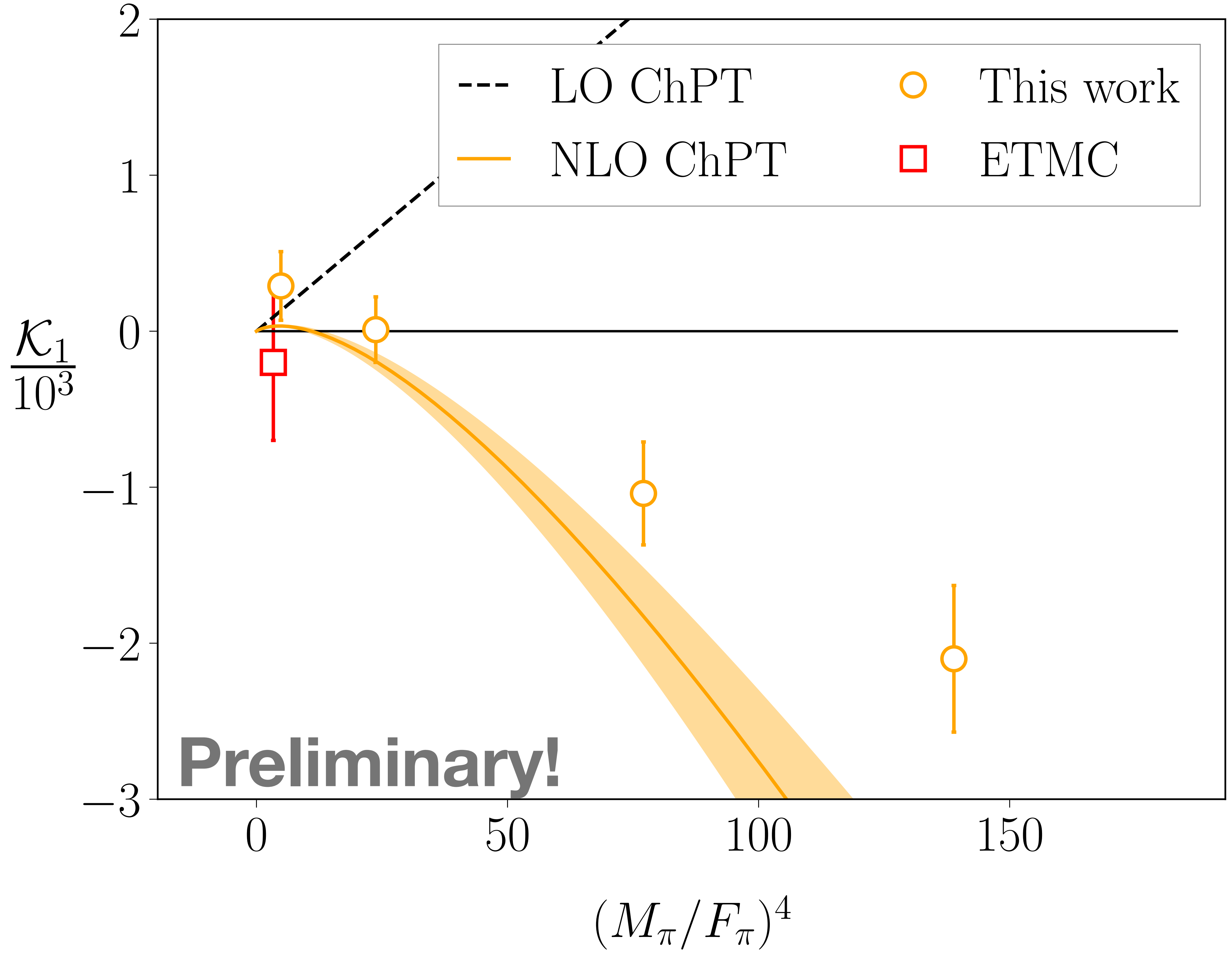
$$\mathcal{K}_{\mathrm{df},3} = \mathcal{K}_0 + \mathcal{K}_1\left(\frac{s-9M_\pi^2}{9M_\pi^2}\right) + \cdots$$

NLO ChPT: [Baeza-Ballesteros, Bijnens, Husek, FRL, Sharpe, Sjö, JHEP 2023] [See talk by M. Sjö]

ETMC: [Fischer, Kostrzewa, Liu, FRL, Ueding, Urbach, EPJC 2021]

This work: [Dawid, Draper, Hanlon, Hörz, Morningstar, FRL, Sharpe, Skinner, JHEP 2023 + on-going work]





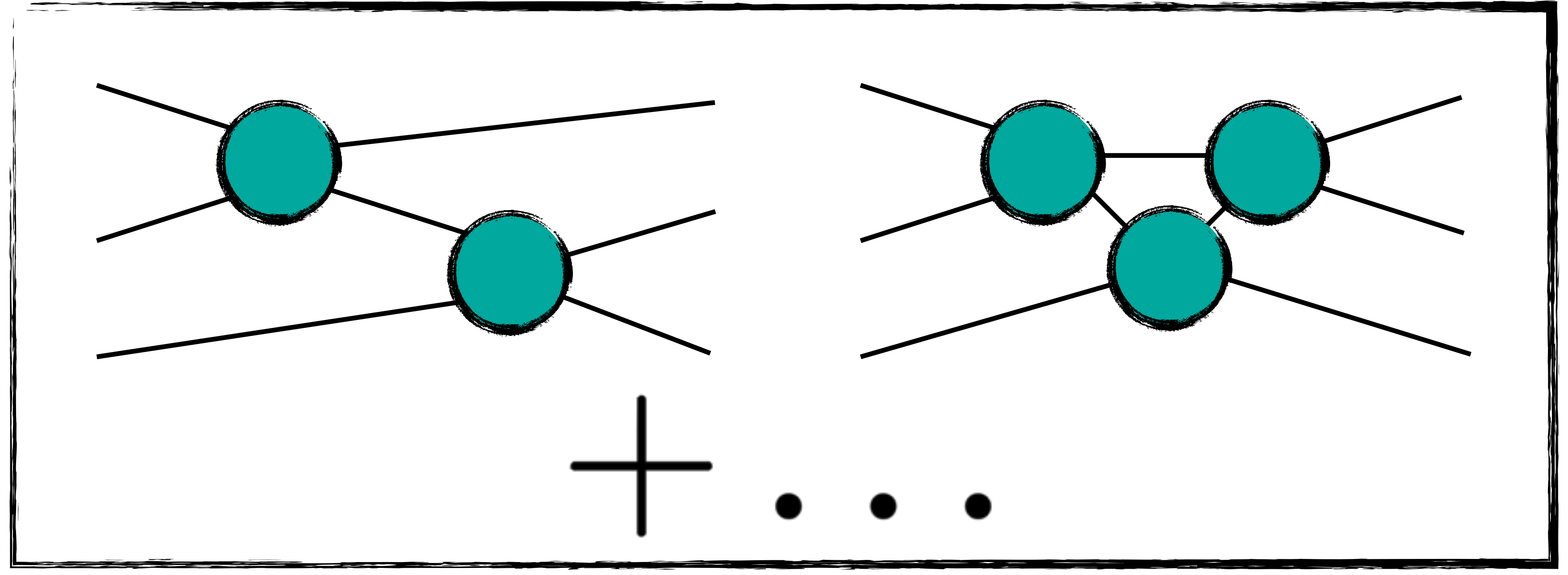
Fernando Romero-López, Uni Bern

Scattering amplitudes

- Physical amplitudes that are consistent with unitary are obtained after solving integral equations:

$$\mathcal{M}_3 = \mathcal{D} + \mathcal{M}_{\mathrm{df},3}$$

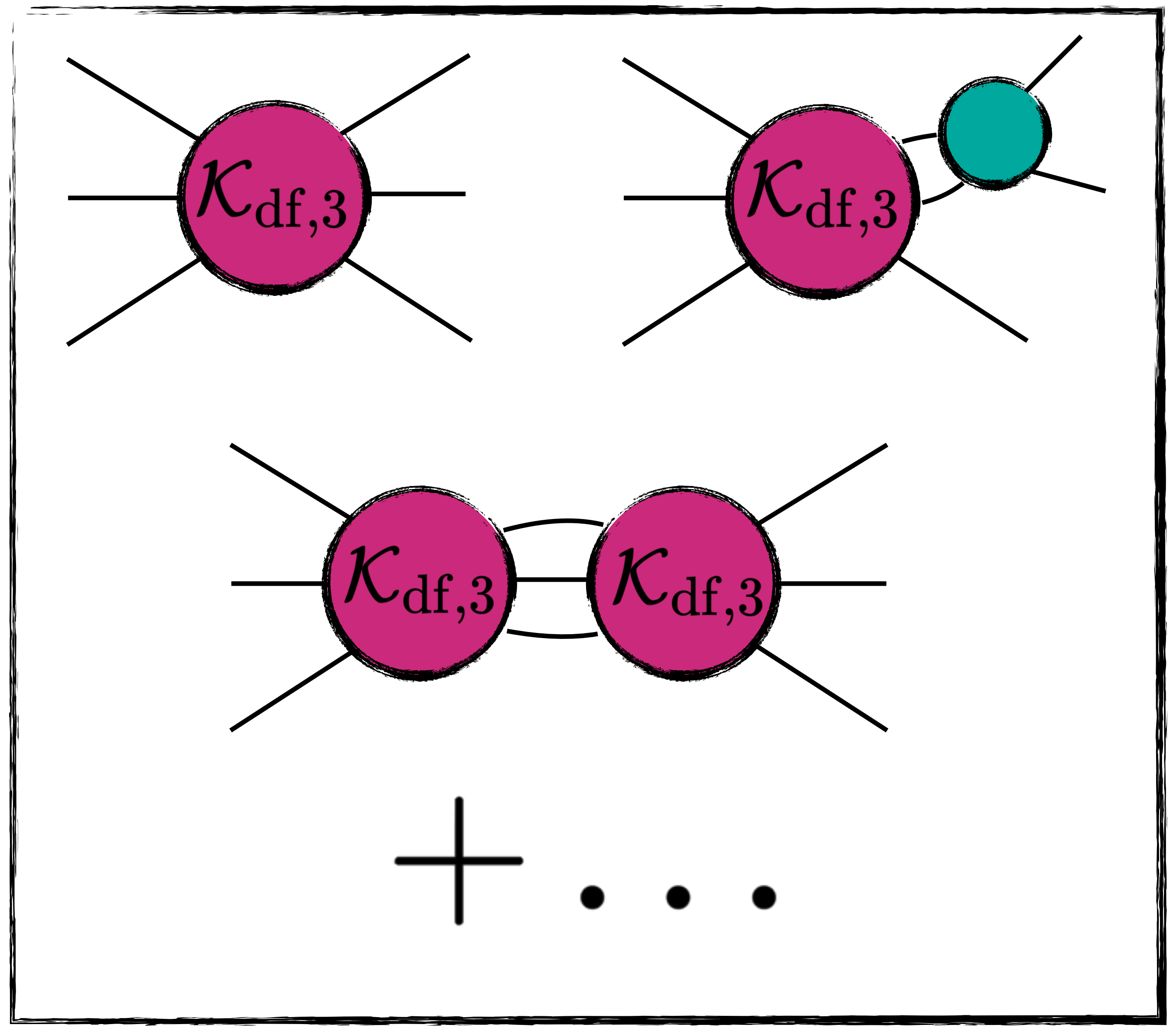
"ladder amplitude"



two-body rescattering

$$\mathcal{D} = -\mathcal{M}_2 G \mathcal{M}_2 - \int \mathcal{M}_2 G \mathcal{D}$$

"divergence-free amplitude"



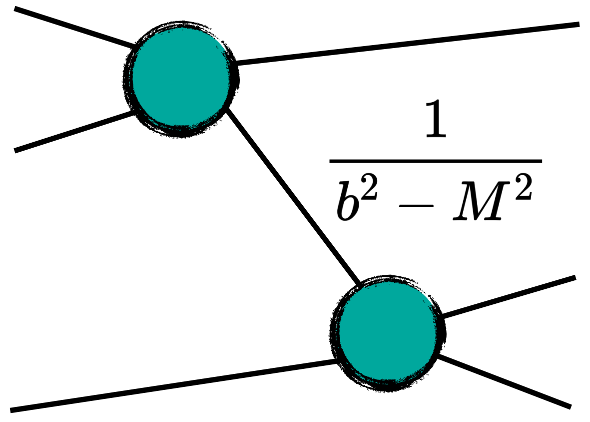
At least one
three-body interaction

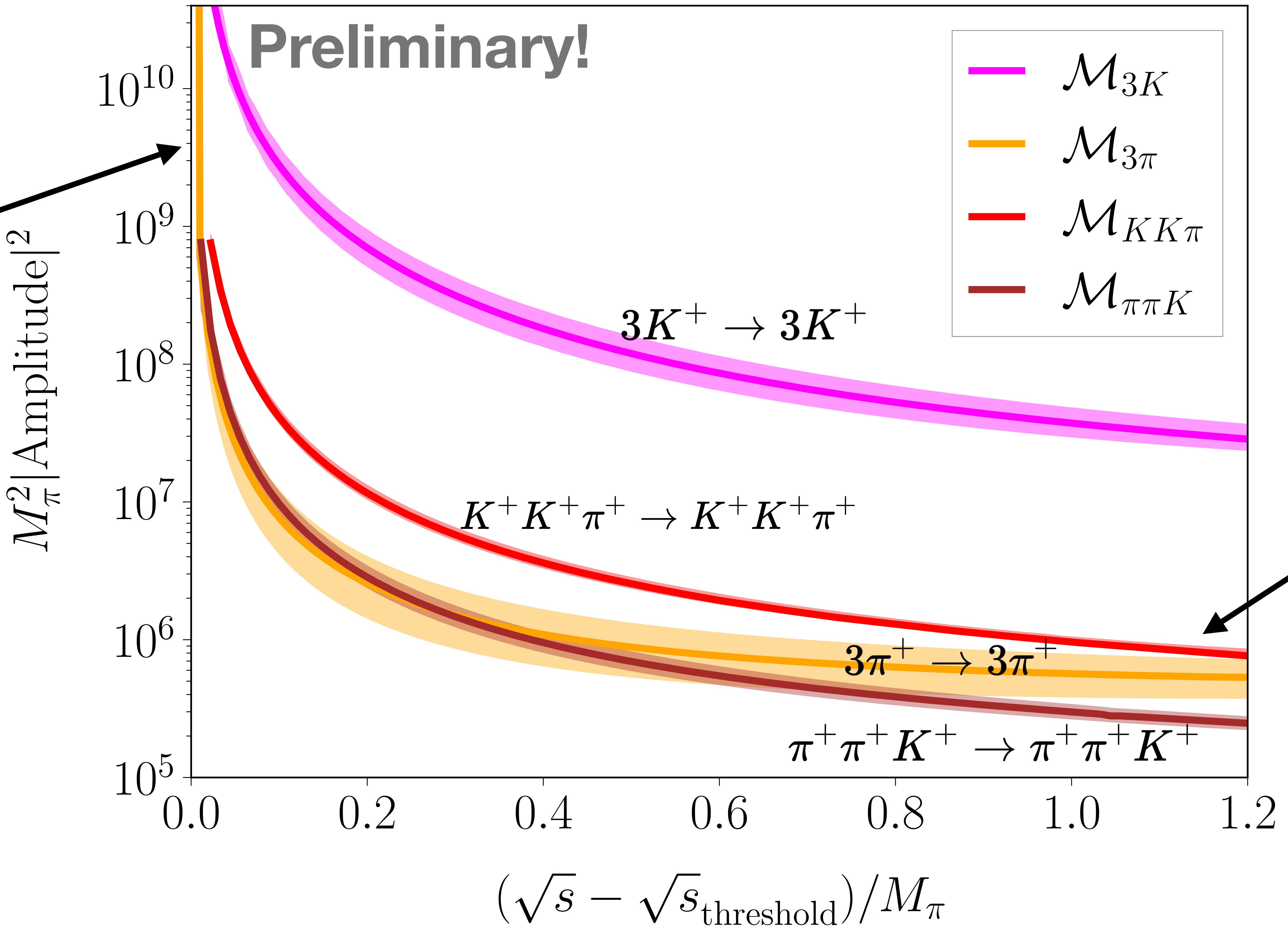
Three-meson amplitudes

Lattice QCD predictions for physical three-meson scattering amplitudes

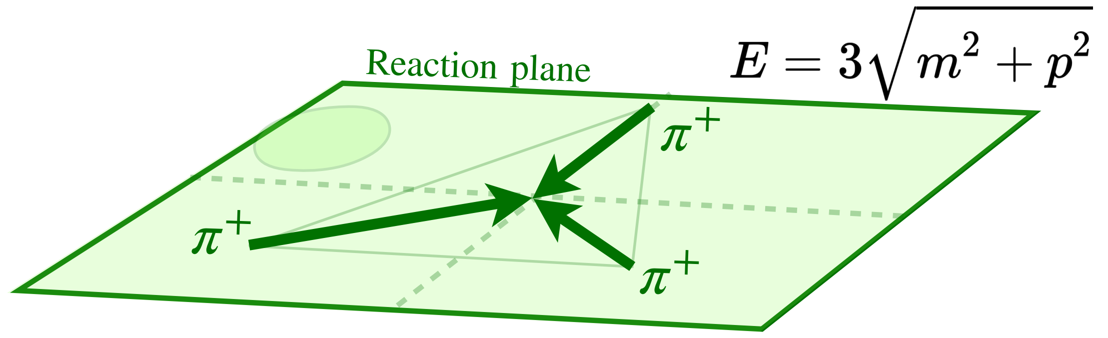
[Dawid, Draper, Hanlon, Hörz, Morningstar, FRL, Sharpe, Skinner, on-going work]

Divergent at threshold





Equilateral kinematic configuration



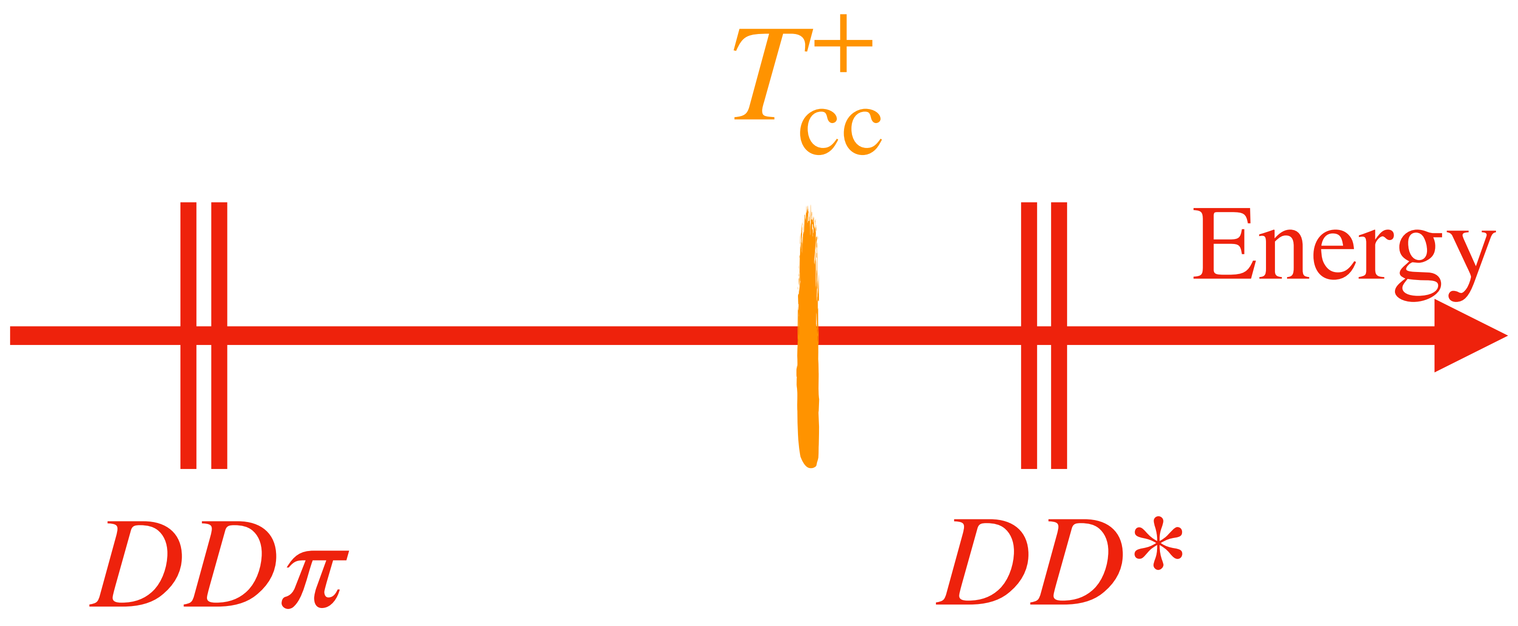
Pion interactions are chirally suppressed

$$M_\pi^2 \mathcal{M}_3 = O(M_\pi^4/F_\pi^4)$$

A three-body description of the T_{cc}

Doubly-charmed tetraquark

Experiment

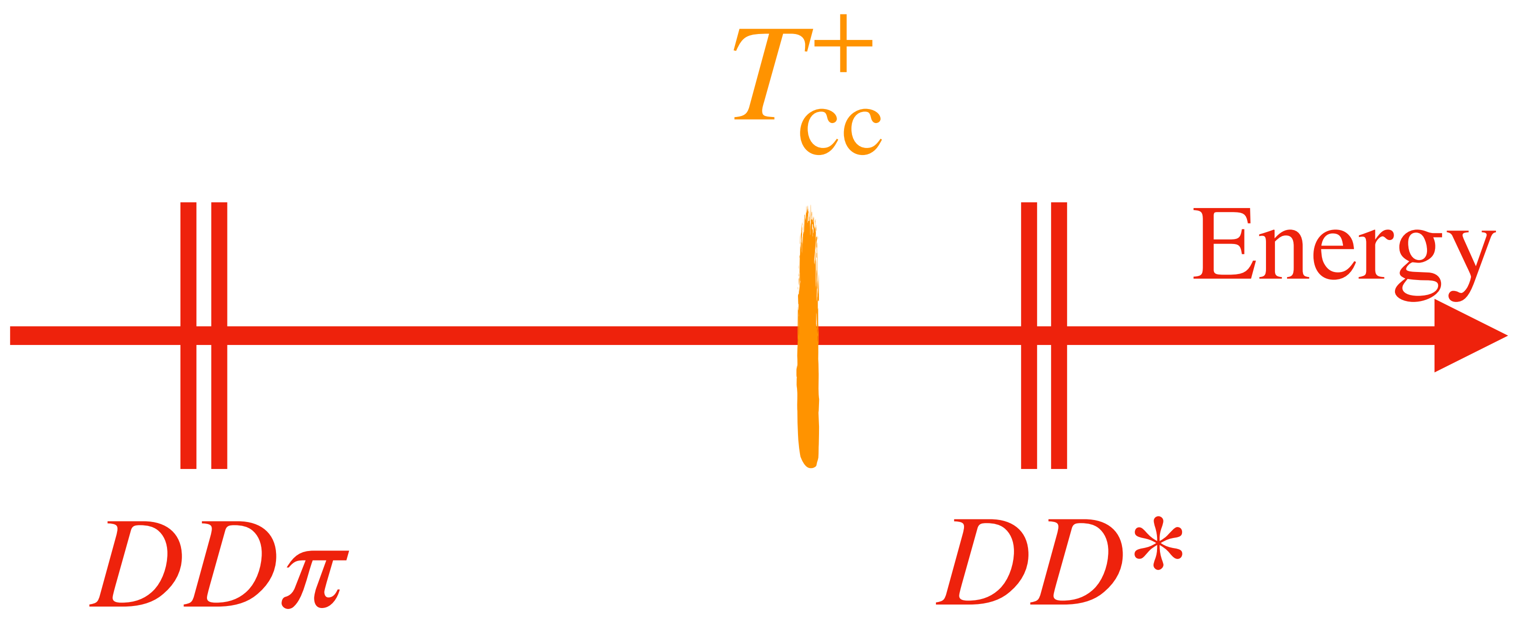


- **For physical quark masses is a three-body resonance**
  $T_{cc} \to DD\pi$

need three-body formalism!

Doubly-charmed tetraquark

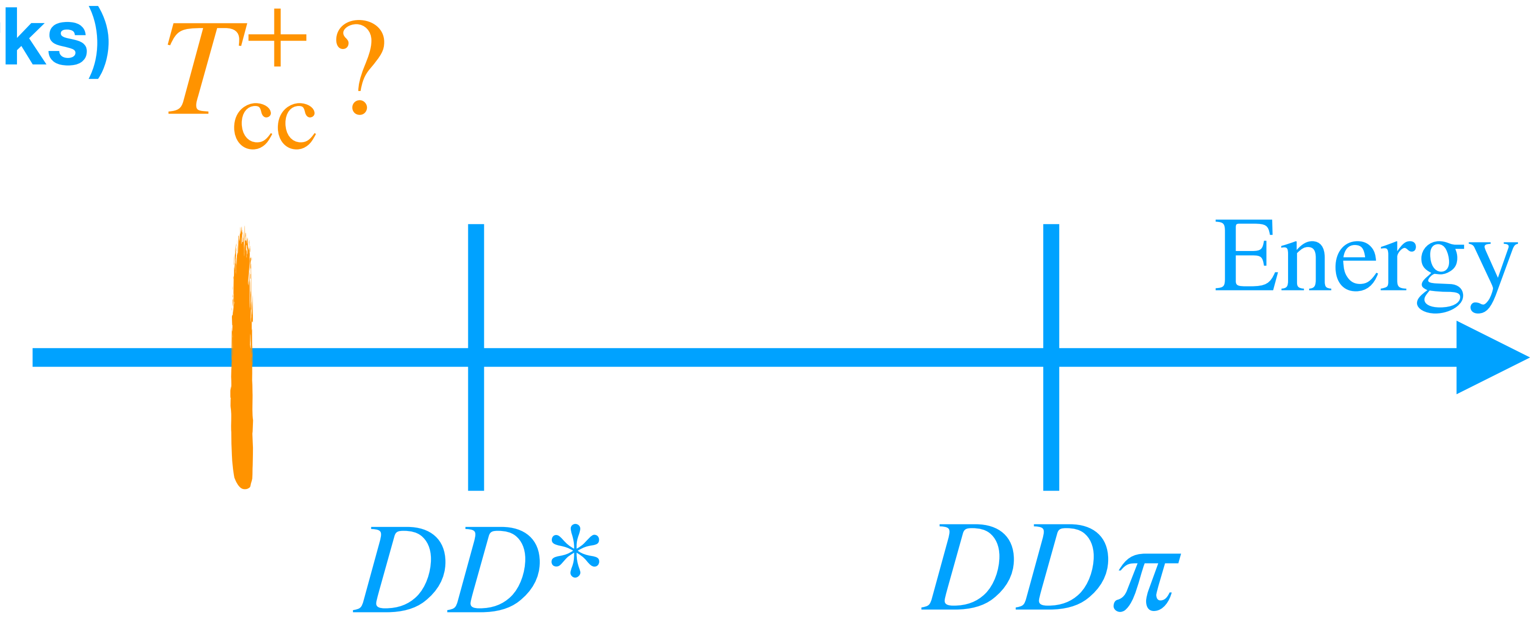
Experiment



- **For physical quark masses is a three-body resonance**
  $T_{cc} \to DD\pi$

need three-body formalism!

N_f=2+1+1 QCD
(heavier quarks)



- **Stable D* at slightly heavier-than-physical quark mases**
  $T_{cc} \to DD^*?$

suitable for the two-body finite-volume formalism?

D-D* scattering

- Several works study the T_{cc} channel in this setup

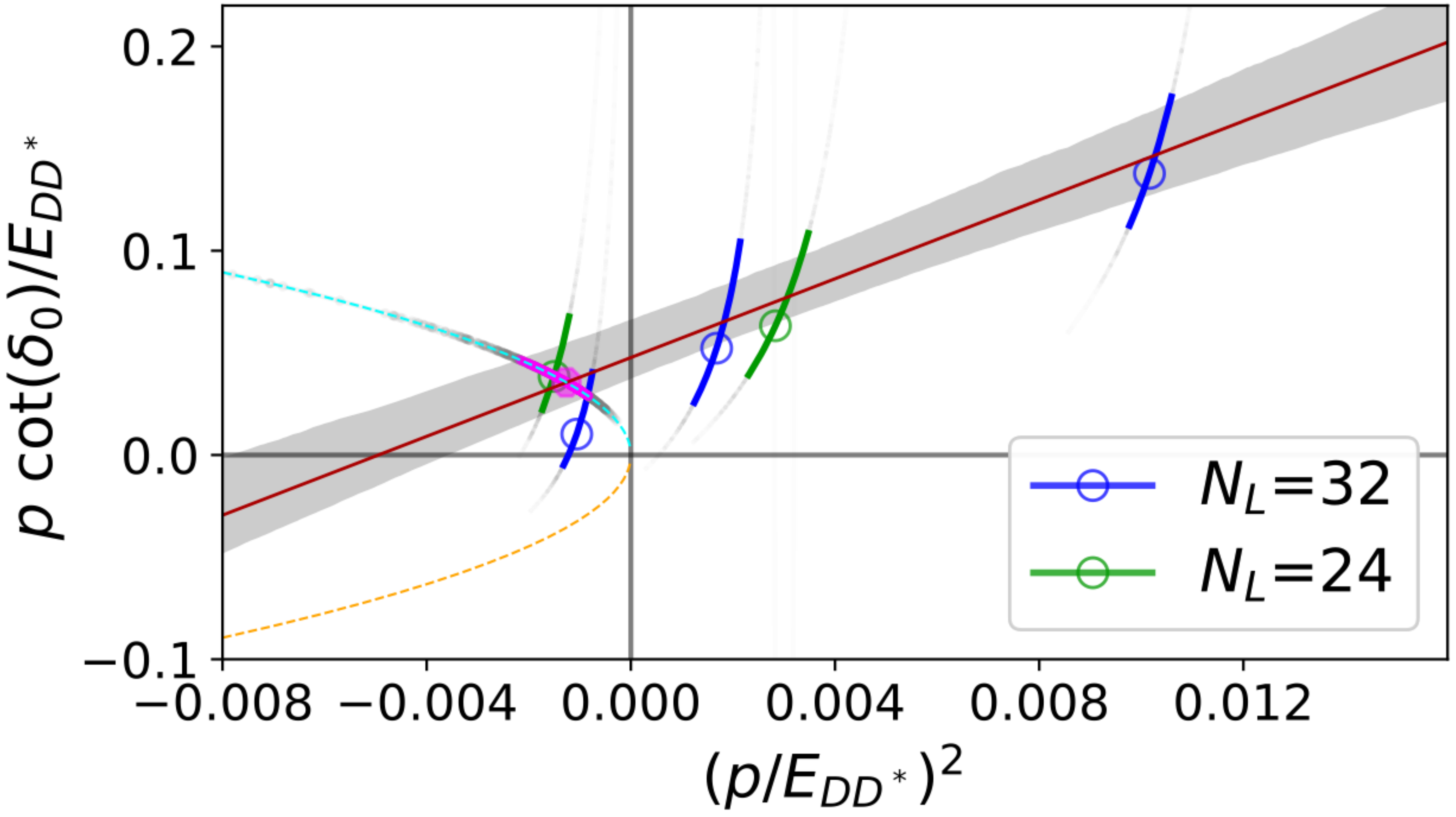
  [Chen et al., 2206.06185] [Lyu et al. (HALQCD), 2302.04505]

  [Padmanath & Prelovsek, 2202.10110]

  [Whyte, Thomas, Wilson, 2405.15741]

  - Signature of virtual bound state?

[Padmanath, Prelovsek, arXiv:2202.10110]



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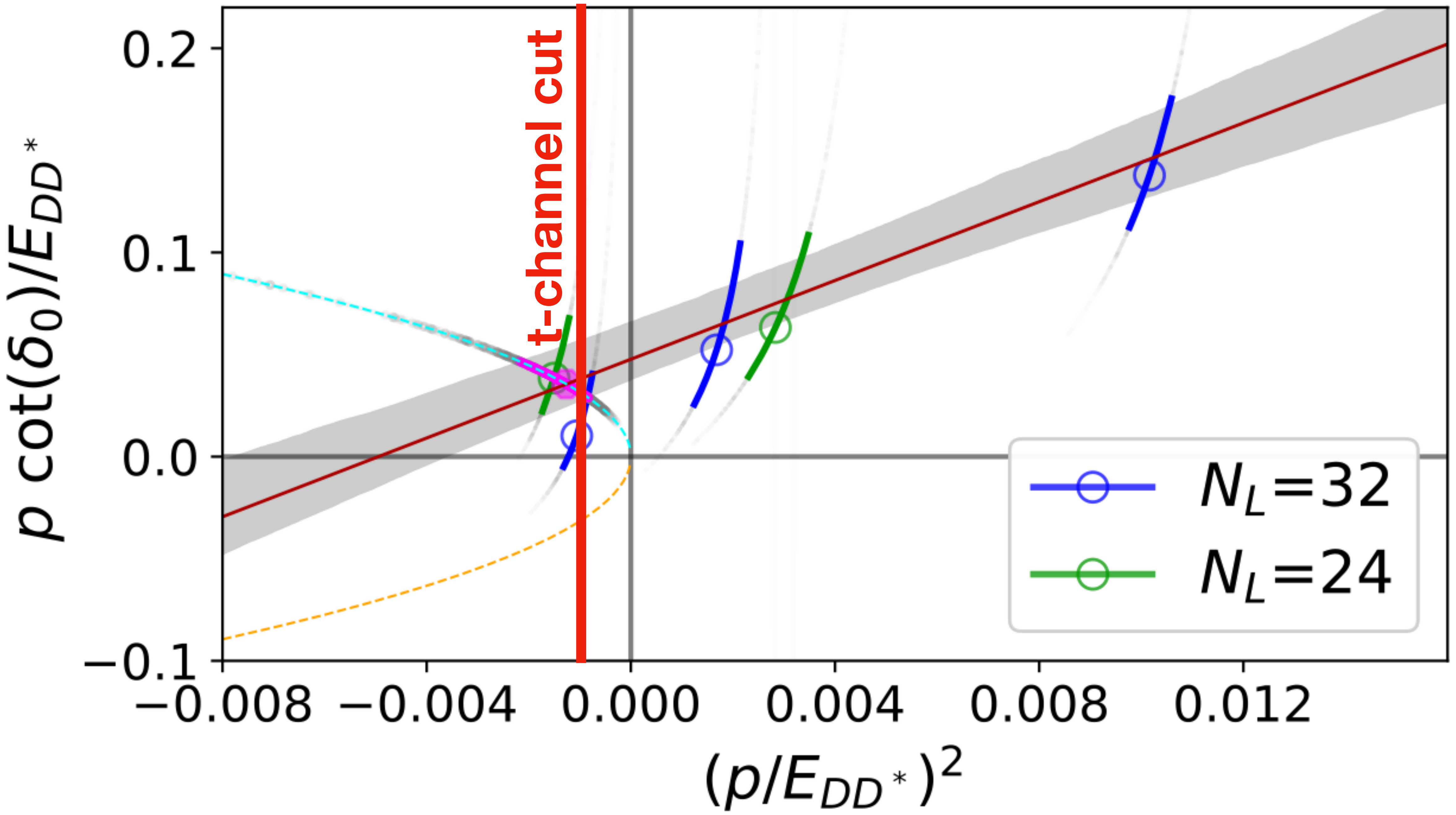
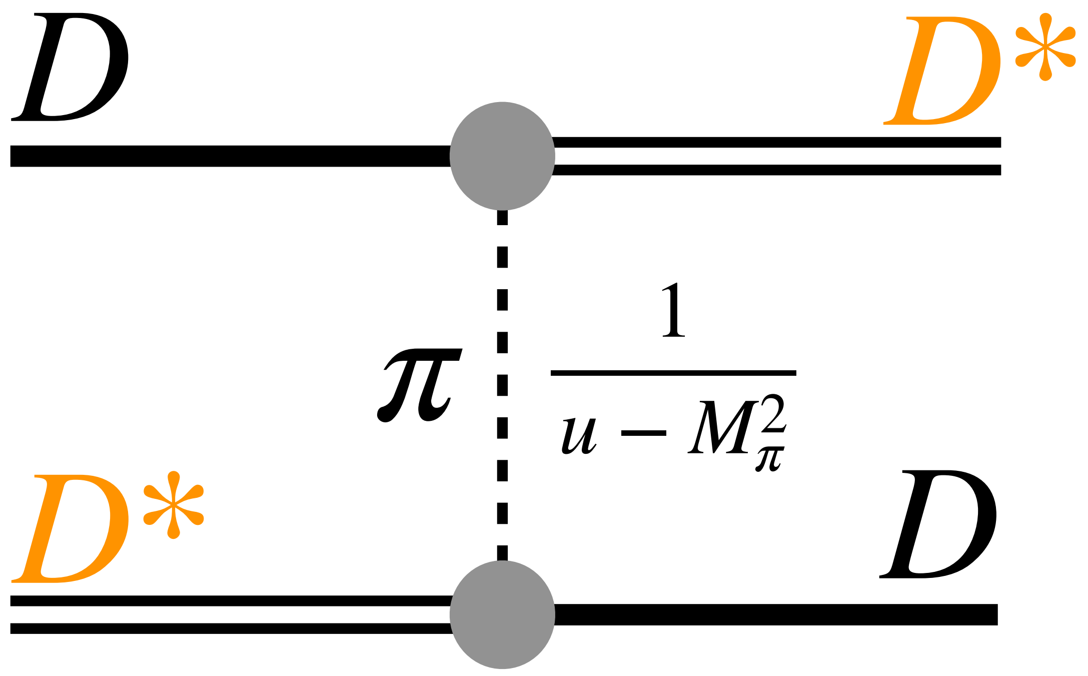
  [Padmanath & Prelovsek, 2202.10110]
  [Whyte, Thomas, Wilson, 2405.15741]
  - Signature of virtual bound state?
  - But two-particle formalism breaks down
    i.e. complex phase shift

[Padmanath, Prelovsek, arXiv:2202.10110]



! one-pion exchange creates non-analytic behavior:



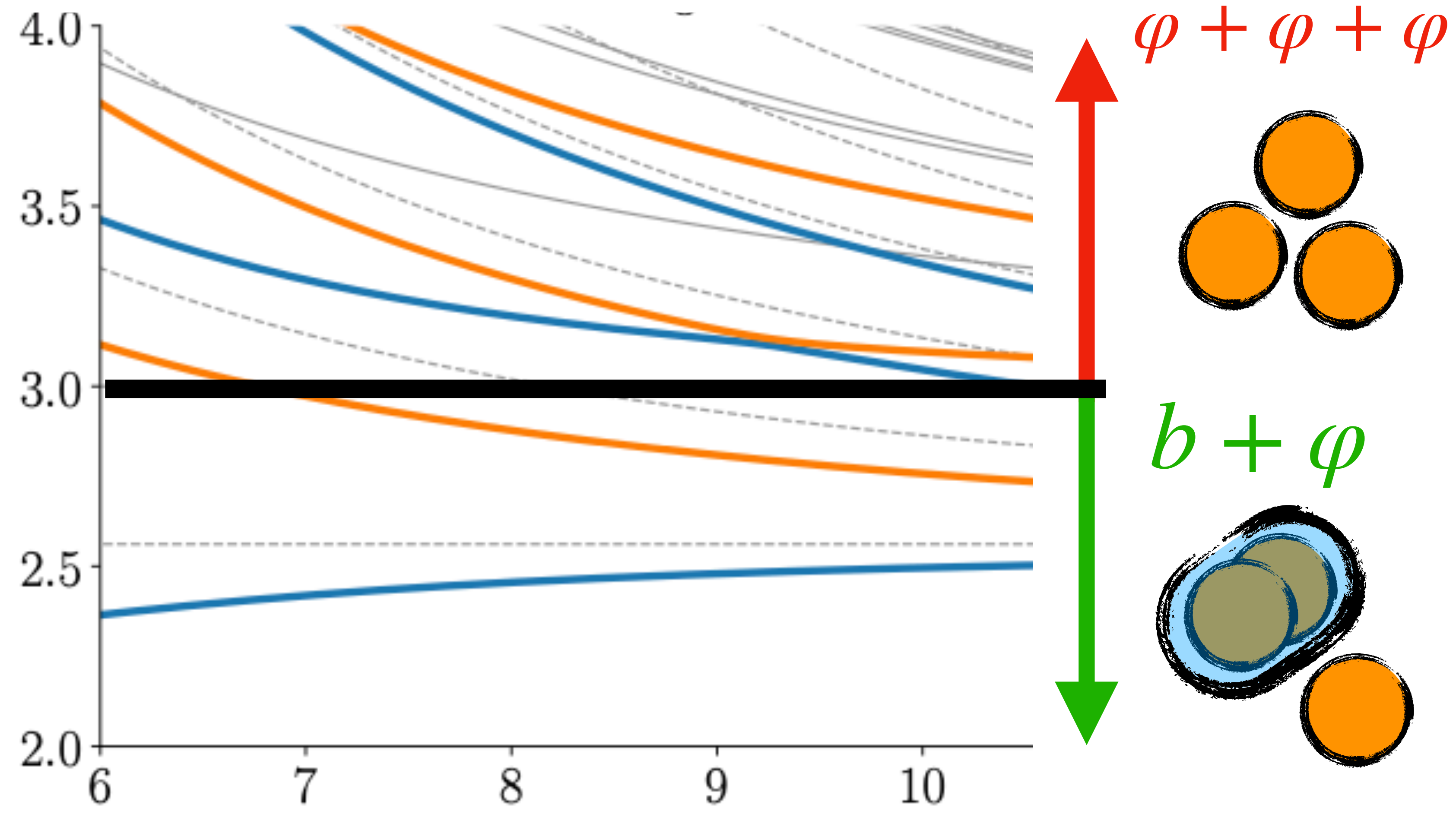
$$u = M_\pi^2, \quad t = 0, \quad s - s_{th} = -M_\pi^2 + (M_D - M_{D^*})^2$$

just 8 MeV below threshold!

A three-body solution

- In the presence of a two-body bound state:
  - Below the three-particle threshold, effective “particle-dimer”

[FRL et al 2302.04505] [Jackura et al 2010.09820]
[Dawid, Islam, Briceño, 2303.04394]
[Briceño, Jackura, Pefkou, FRL 2402.12167]



[FRL, Sharpe, Blanton, Briceño, Hansen 1908.02411]

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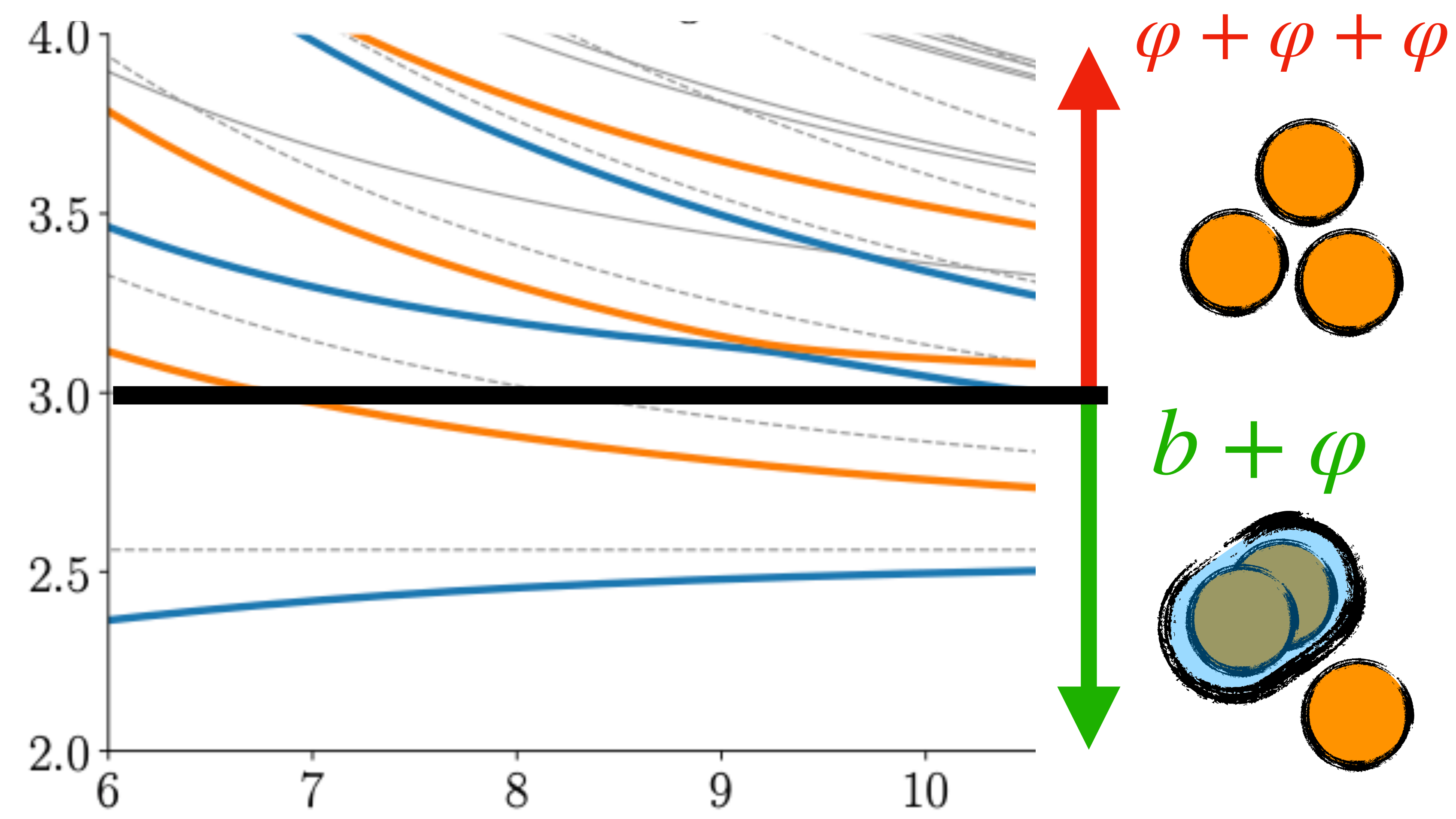
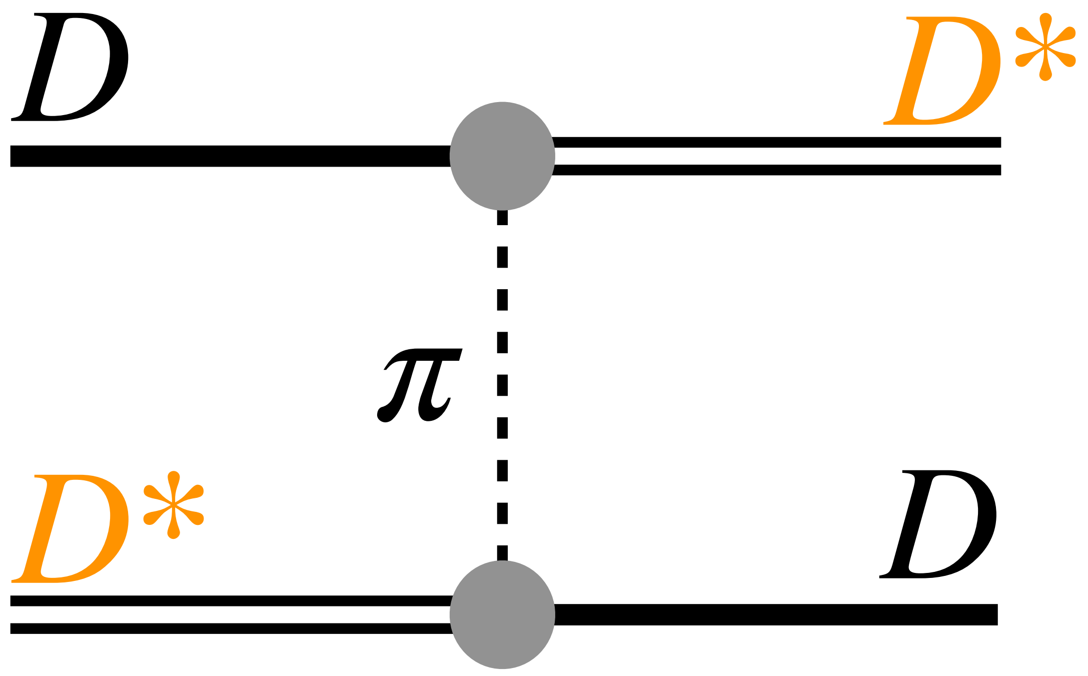
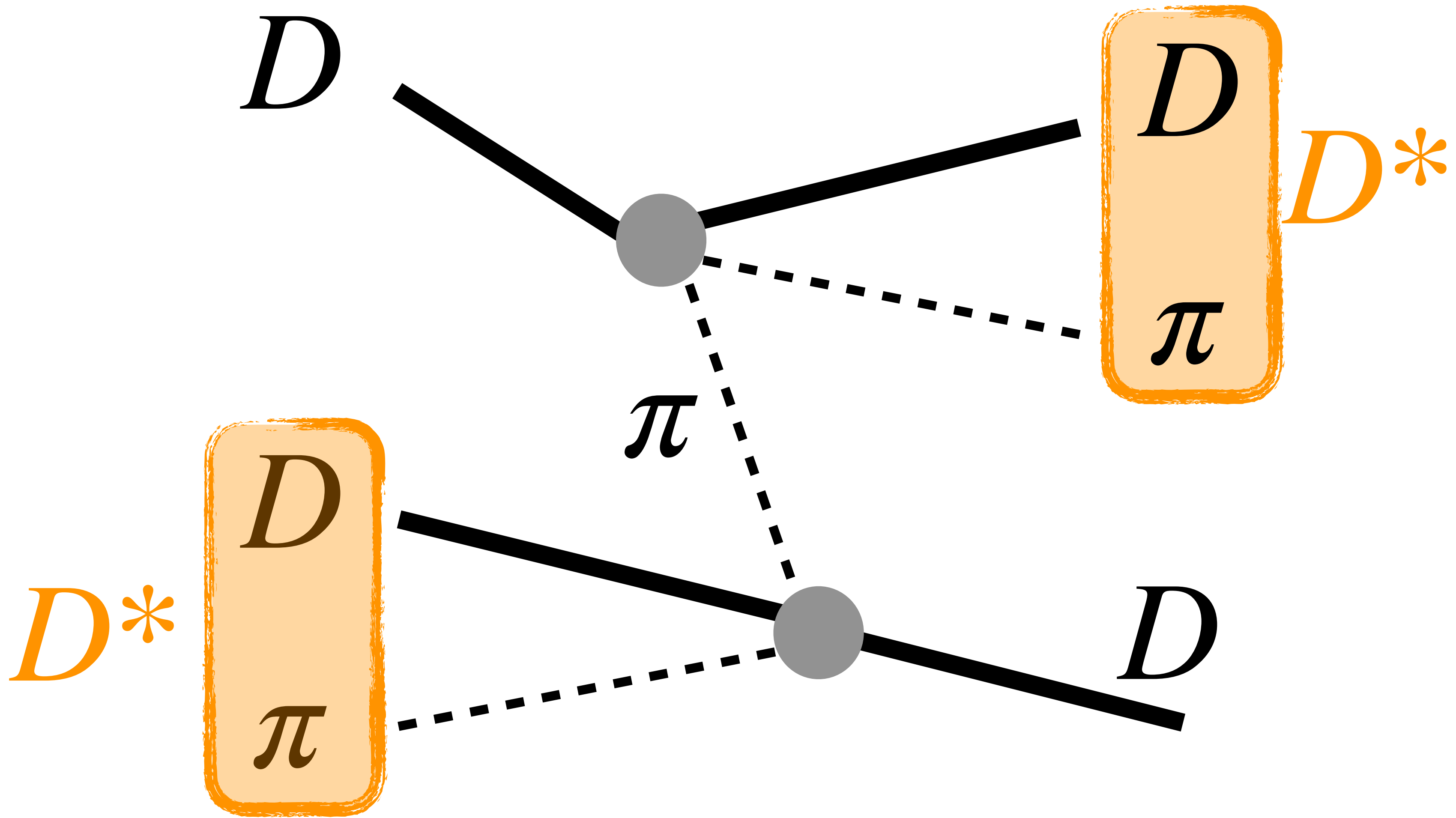


- This solves the left-hand cut problem:
  - Finite-volume effects from one-pion exchange naturally incorporated

    Other approaches: [Du et al (2408.09375), Abolnikov et al. (2407.04649), Bubna et al. (2402.12985),
    Meng et al. (2312.01930), Raposo, Hansen (2311.18793)]



[Hansen, FRL, Sharpe, arXiv:2401.06609]

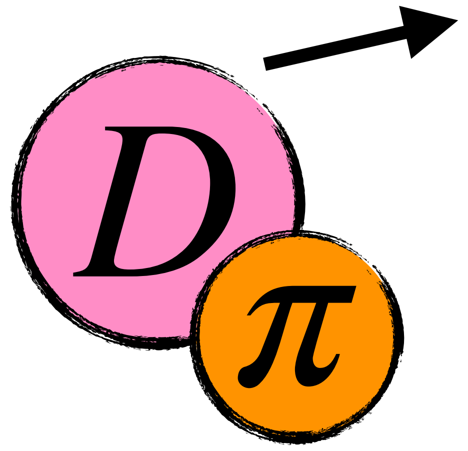


The strategy for the T_{cc}

[Hansen, FRL, Sharpe, arXiv:2401.06609]

Two-meson spectra

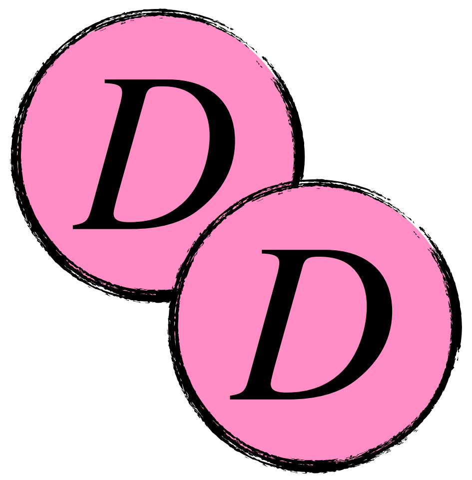
$I_{D\pi} = 1/2$



D^* as a bound state or resonance

s and p waves

$I_{DD} = 1$



s wave

The strategy for the T_{cc}

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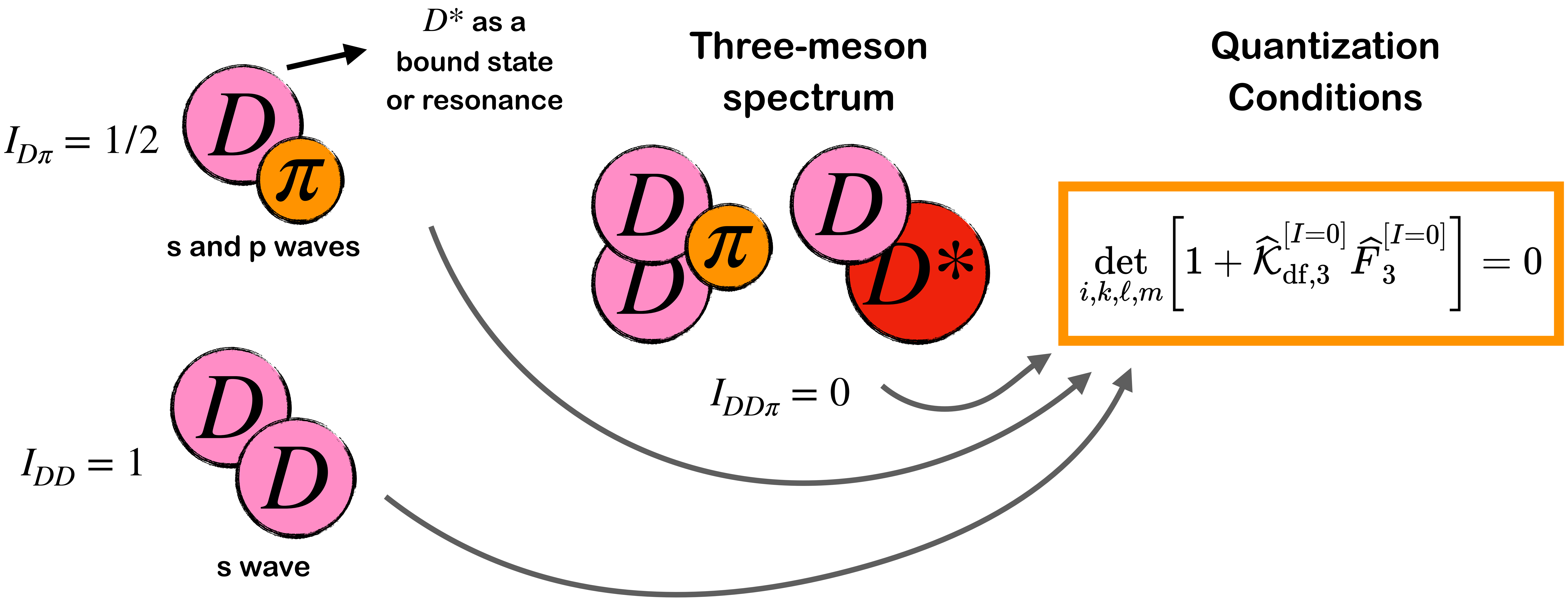

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s and p waves

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s wave

Three-meson spectrum

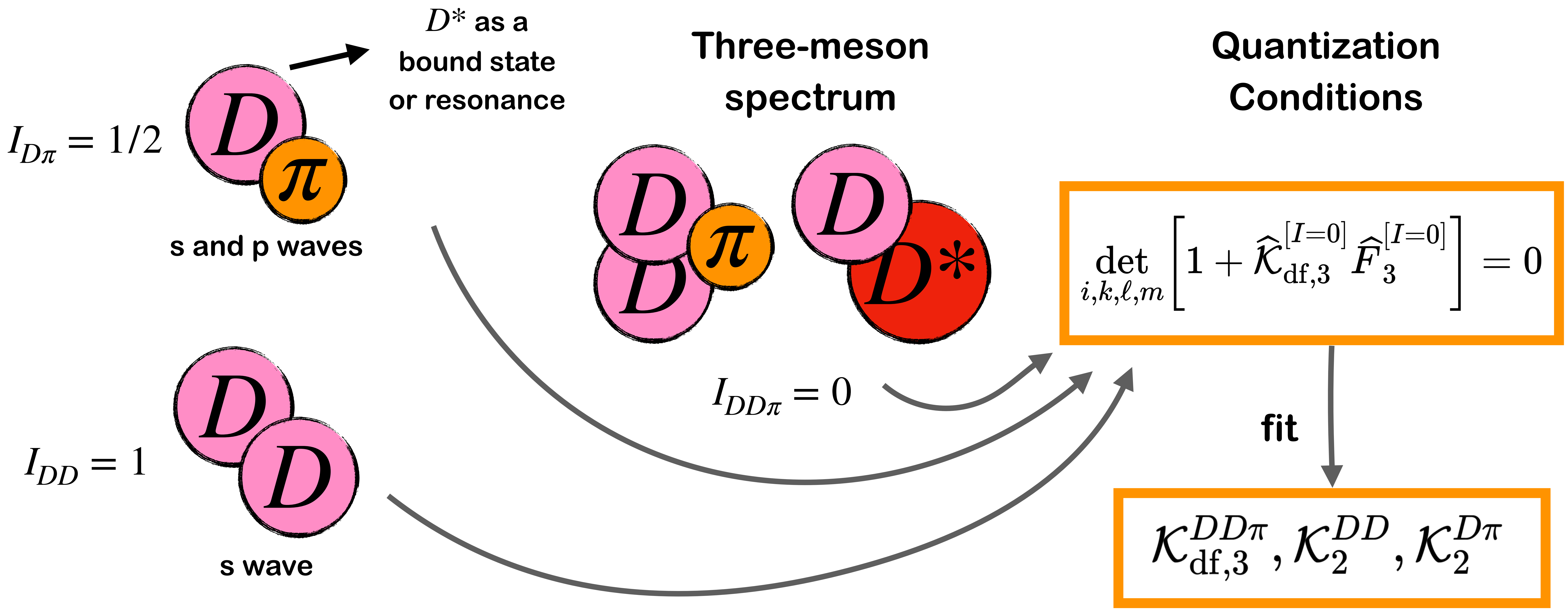
$I_{DD\pi} = 0$

Quantization Conditions

$$\boxed{\det_{i,k,\ell,m}\left[1+\widehat{\mathcal{K}}_{\mathrm{df},3}^{[I=0]}\widehat{F}_3^{[I=0]}\right]=0}$$

The strategy for the T_{cc}

[Hansen, FRL, Sharpe, arXiv:2401.06609]



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fit

$$\boxed{\mathcal{K}_{\mathrm{df},3}^{DD\pi}, \mathcal{K}_2^{DD}, \mathcal{K}_2^{D\pi}}$$

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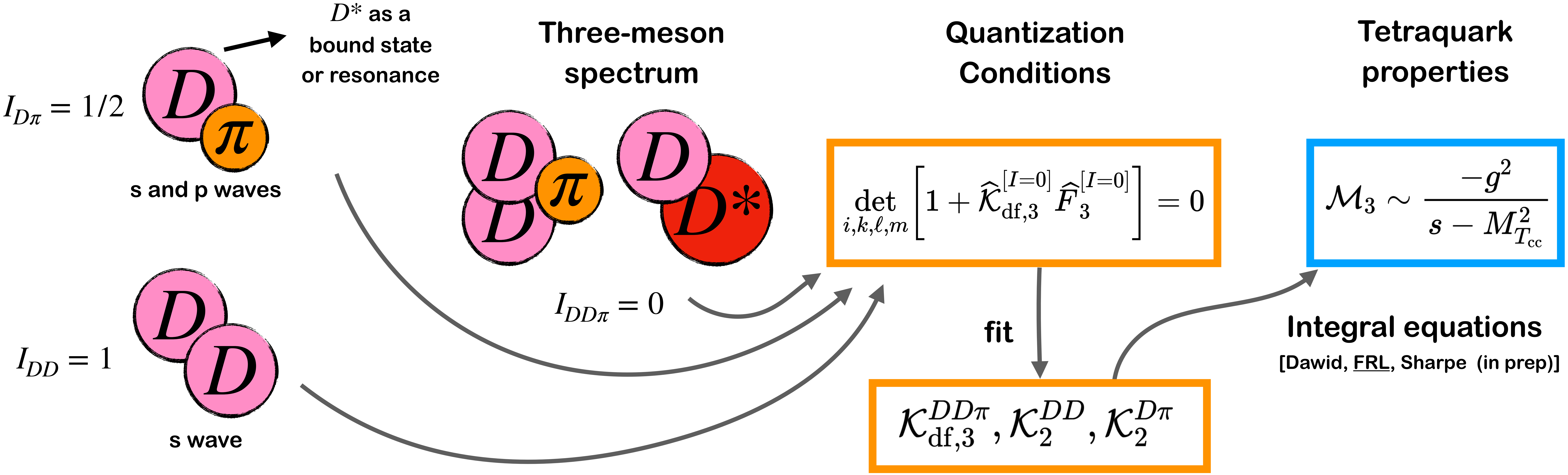
[Hansen, FRL, Sharpe, arXiv:2401.06609]



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fit

$$\boxed{\mathcal{K}_{\text{df},3}^{DD\pi}, \mathcal{K}_2^{DD}, \mathcal{K}_2^{D\pi}}$$

Tetraquark properties

$$\boxed{\mathcal{M}_3 \sim \frac{-g^2}{s - M_{T_{cc}}^2}}$$

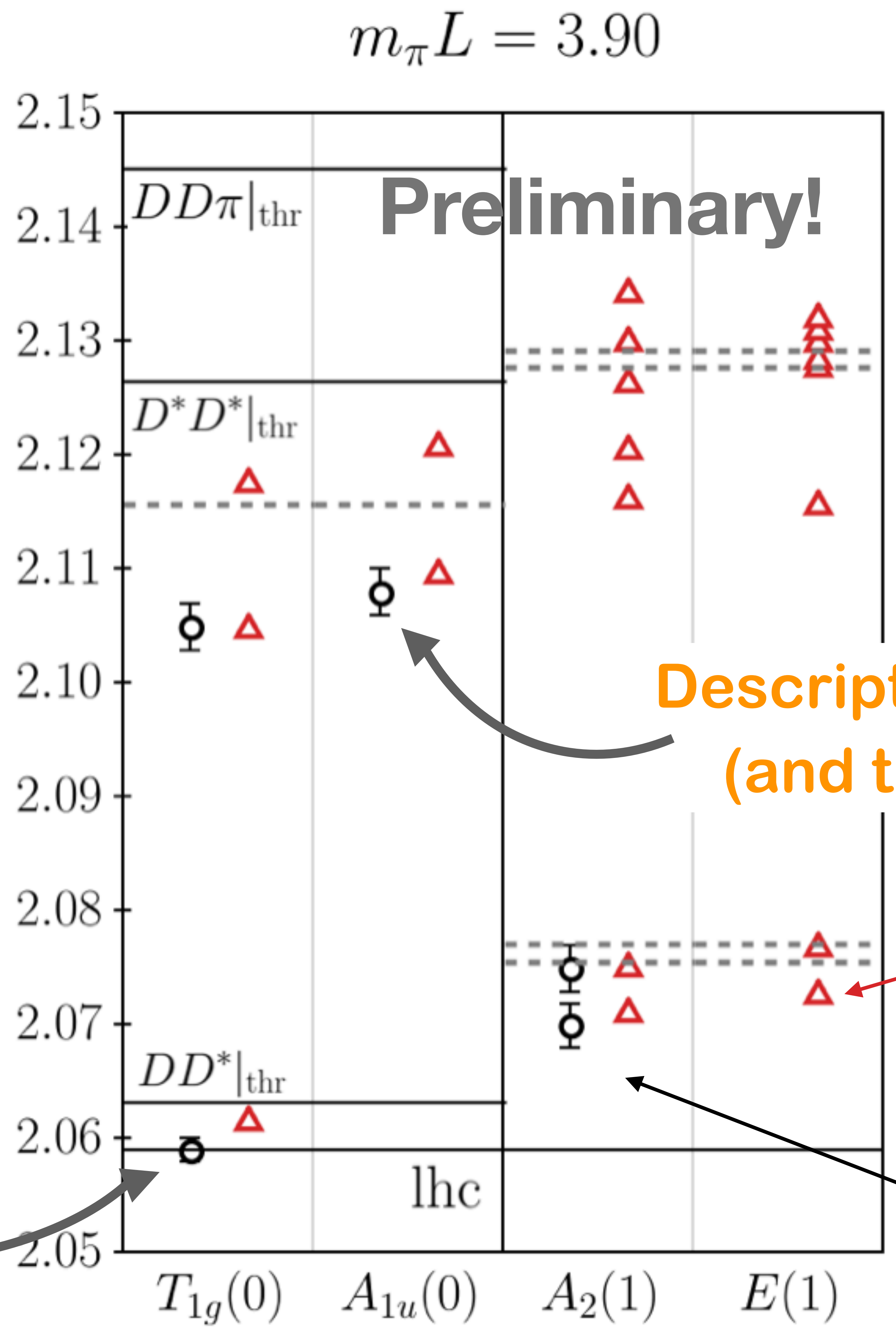
Integral equations

[Dawid, FRL, Sharpe (in prep)]

Analyzing D-D* data

- Published data only provides DD* energies
  [Padmanath, Prelovsek, 2202.10110]
- Fix Dπ and DD interactions with "educated guesses"
  - HChPT and lattice results
  - Neglect DD interactions
- Only "free" parameter in the three-body K matrix

$$\mathcal{K}_{\mathrm{df},3} = \mathcal{K}_E (p_\pi - p'_\pi)^2$$



Description of several irreps (and thus partial waves)

QC3

Lattice QCD

Finite-volume energies near the left-hand cut

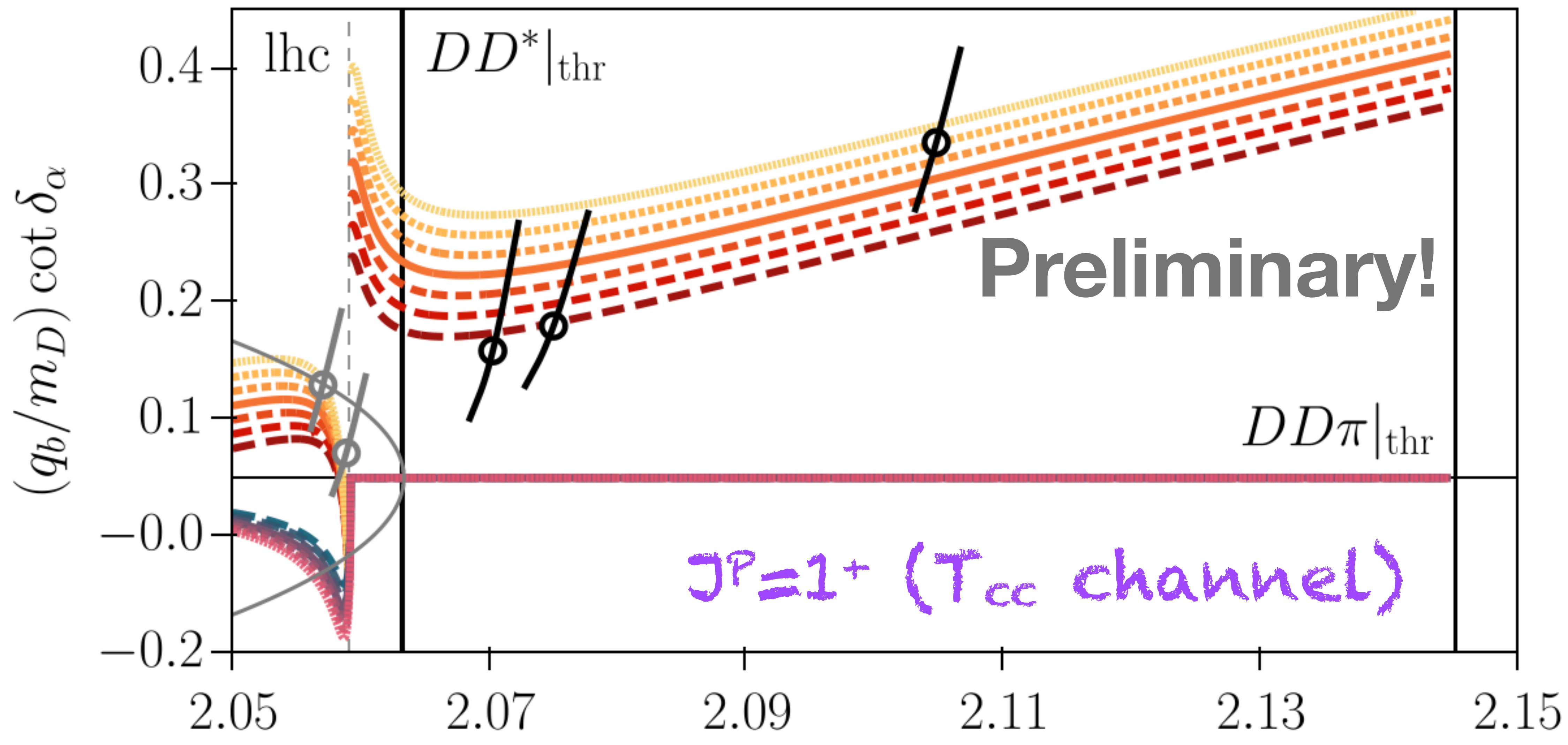
[S. Dawid, FRL, S. Sharpe, (in prep)]

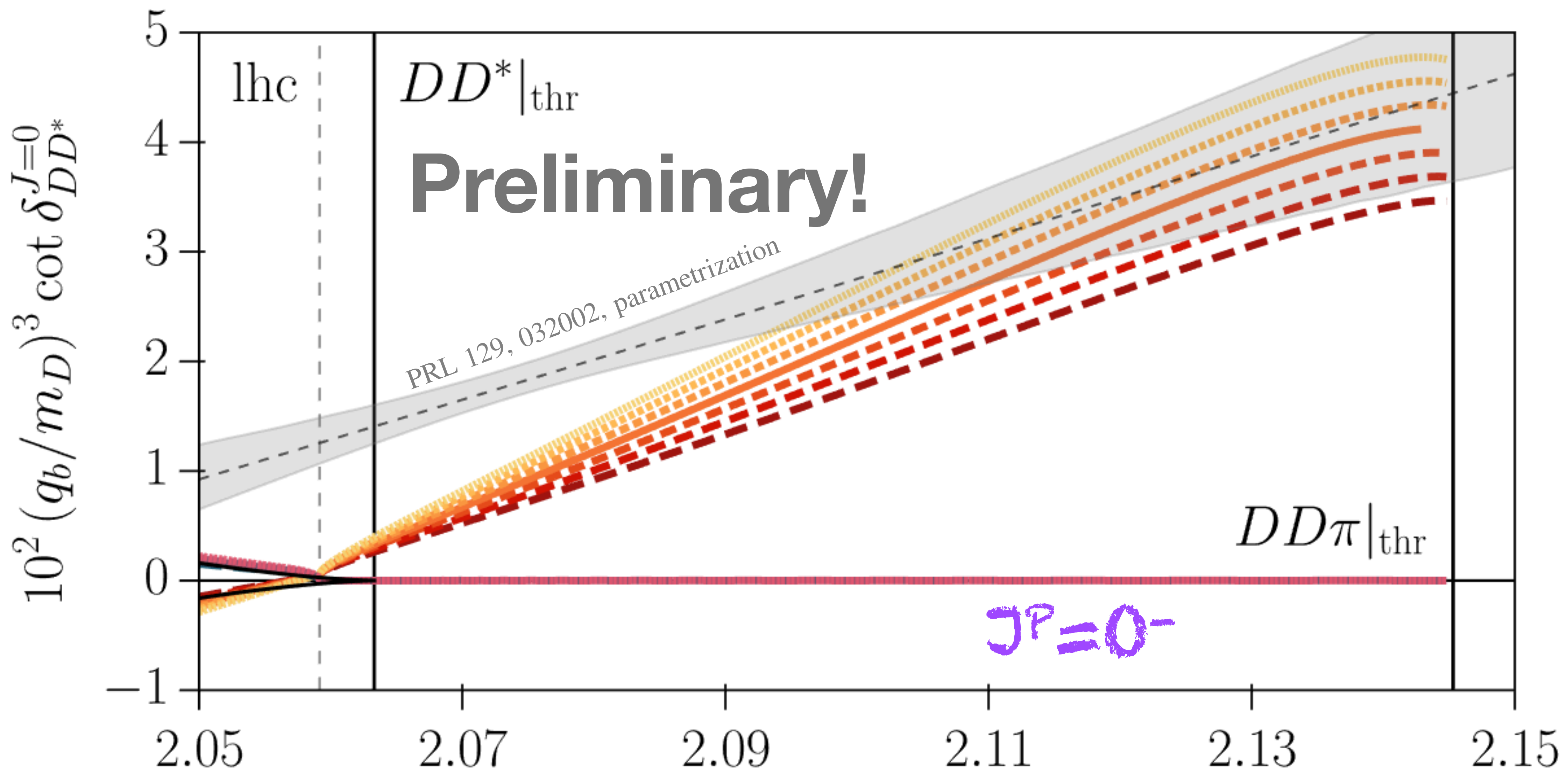
Analyzing D-D* data

- With simple parametrizations we are able to reproduce lattice QCD energies.

[S. Dawid, FRL, S. Sharpe, (in prep)] Data from: [Padmanath, Prelovsek, arXiv:2202.10110]

[More details in parallel by S. Dawid (Tuesday)]





- Still lots of modeling, and need a genuine three-body study of the T_{cc} !

Summary and Outlook

Conclusion & Outlook

- Three-hadron dynamics is an important frontier in lattice QCD spectroscopy
    - Resonances, exotics, three-baryon forces, electroweak decays
- Constraints on three-meson scattering amplitudes:
    - Three-hadron at maximal isospin at the physical point
    - Evidence for “contact” three-hadron force $(3K)$
    - Already some pioneering three-body resonances
      [Yan et al (2407.16659), Garofalo et al. (2211.05605), Mai et al. (2107.03973)]
- Three-meson dynamics is essential to investigate the T_{cc}
    - Solves the issue of the left-hand cut
    - Allows for a smooth transition from stable to unstable D*
- Next steps involve other three-pion isospin channels, $DD\pi$ systems and systems with baryons
    - Needs formalism developments and lattice QCD applications

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Thanks!

References

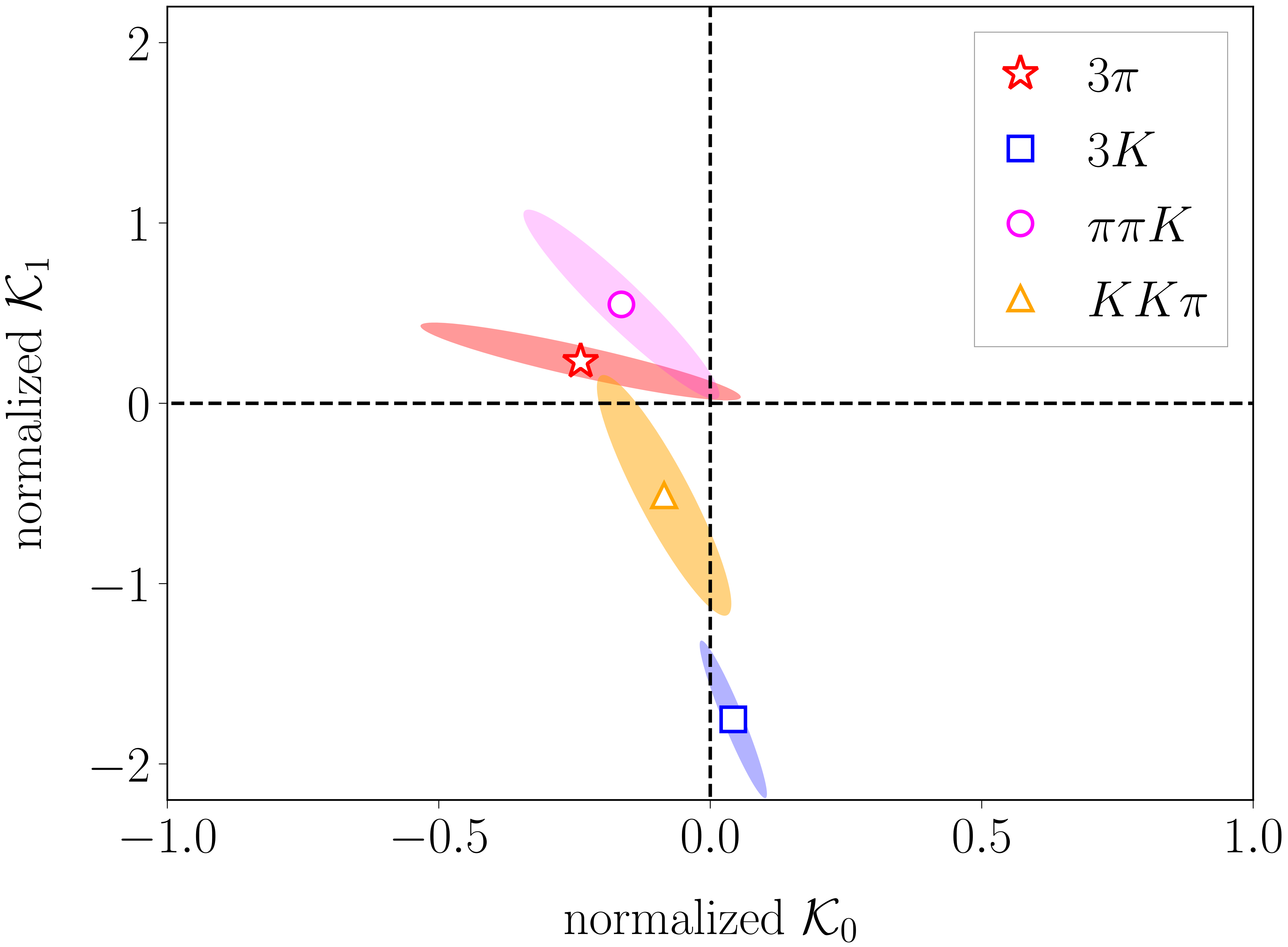
- Interactions of two and three mesons including higher partial waves from lattice QCD
  [Blanton, Hanlon, Hörz, Morningstar, FRL, Sharpe, arXiv:2106.05590]
- Interactions of πK, $\pi\pi$K and KKπ systems at maximal isospin from lattice QCD
  [Draper, Hanlon, Hörz, Morningstar, FRL, Sharpe, arXiv:2302.13587]
- Two- and three-meson scattering amplitudes with physical quark masses from lattice QCD
  [Draper, Dawid, Hanlon, Hörz, Morningstar, FRL, Skinner, Sharpe (in prep)]

- The isospin-3 three-particle K matrix at NLO in ChPT
  [Baeza-Ballesteros, Bijnens, Husek, FRL, Sjö, Sharpe, arXiv:2303.13206]
- The three-pion K matrix at NLO in ChPT
  [Baeza-Ballesteros, Bijnens, Husek, FRL, Sjö, Sharpe, arXiv:2401.14293]

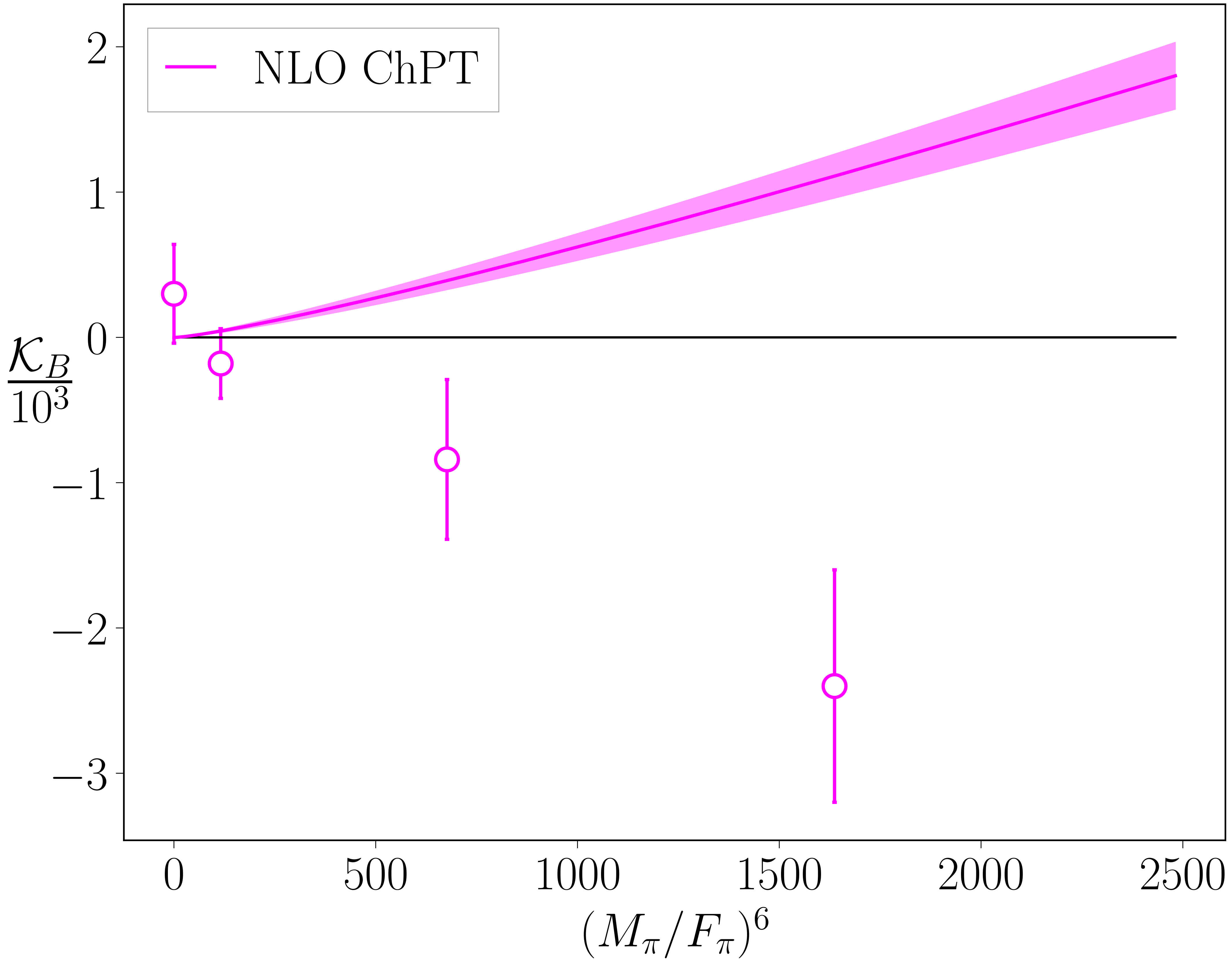
- Incorporating DDπ effects and left-hand cuts in lattice QCD studies of the $T_{cc}(3875)$
  [Hansen, FRL, Sharpe, arXiv:2401.06609]
- A three-body study of the $T_{cc}(3875)$
  [Dawid, FRL, Sharpe, (in prep)]

Back-up

Significance of Kdf3



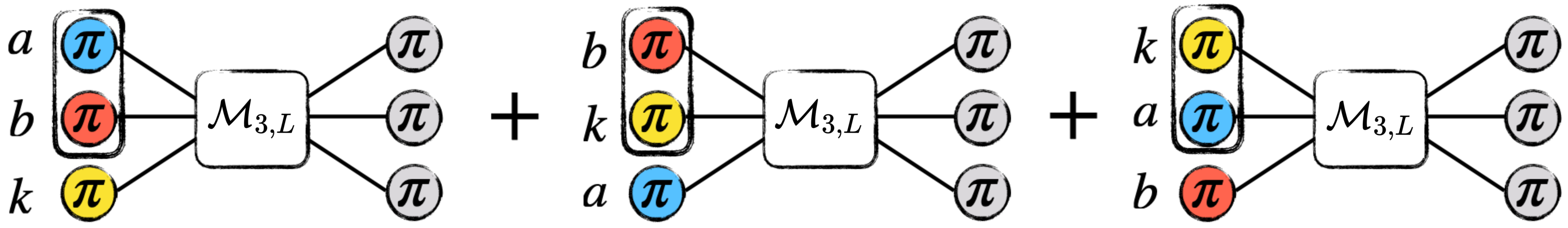
Comparison to ChPT



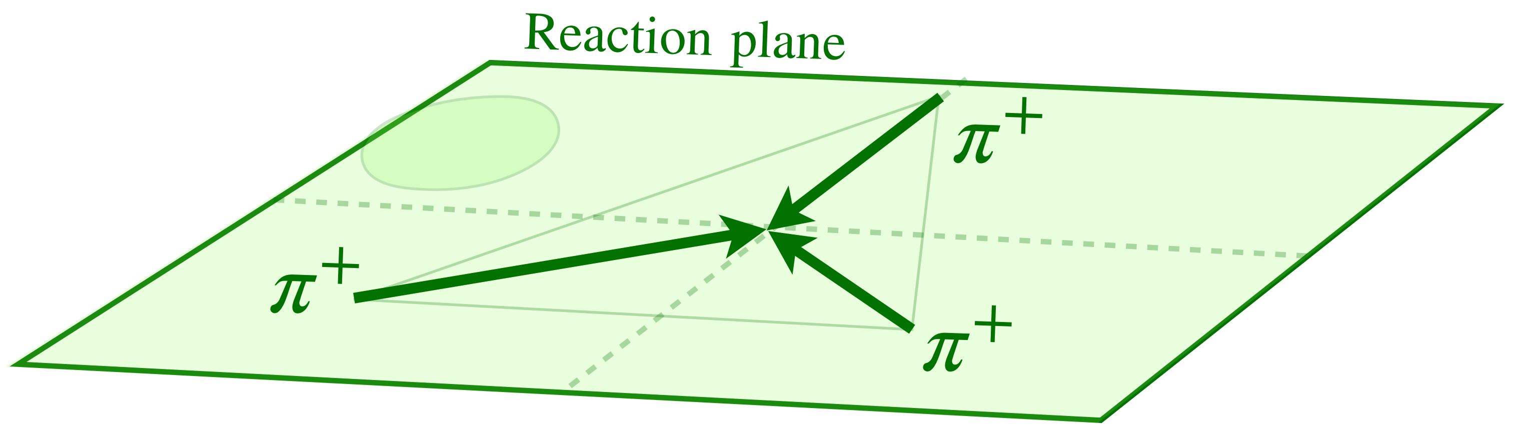
Kinematic configuration

- Symmetrize (each particle gets a turn to be the spectator)

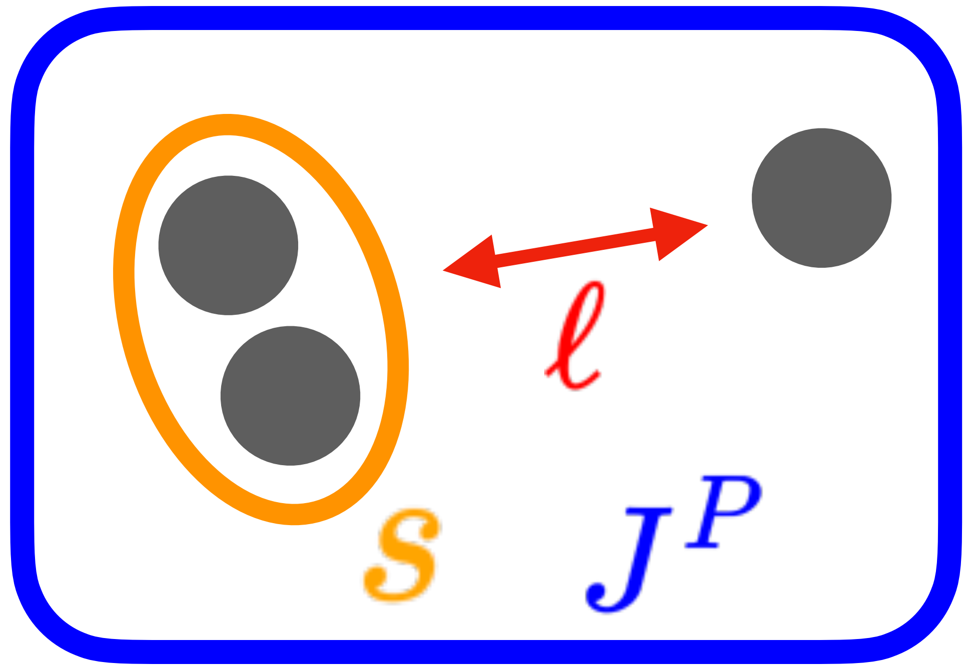
$$\mathcal{M}_{3,L}(P) \equiv \mathcal{S}\left[\mathcal{M}_{3,L}^{(u,u)}(P)\right] =$$



- Equilateral kinematic configuration $E = 3\sqrt{m^2 + p^2}$

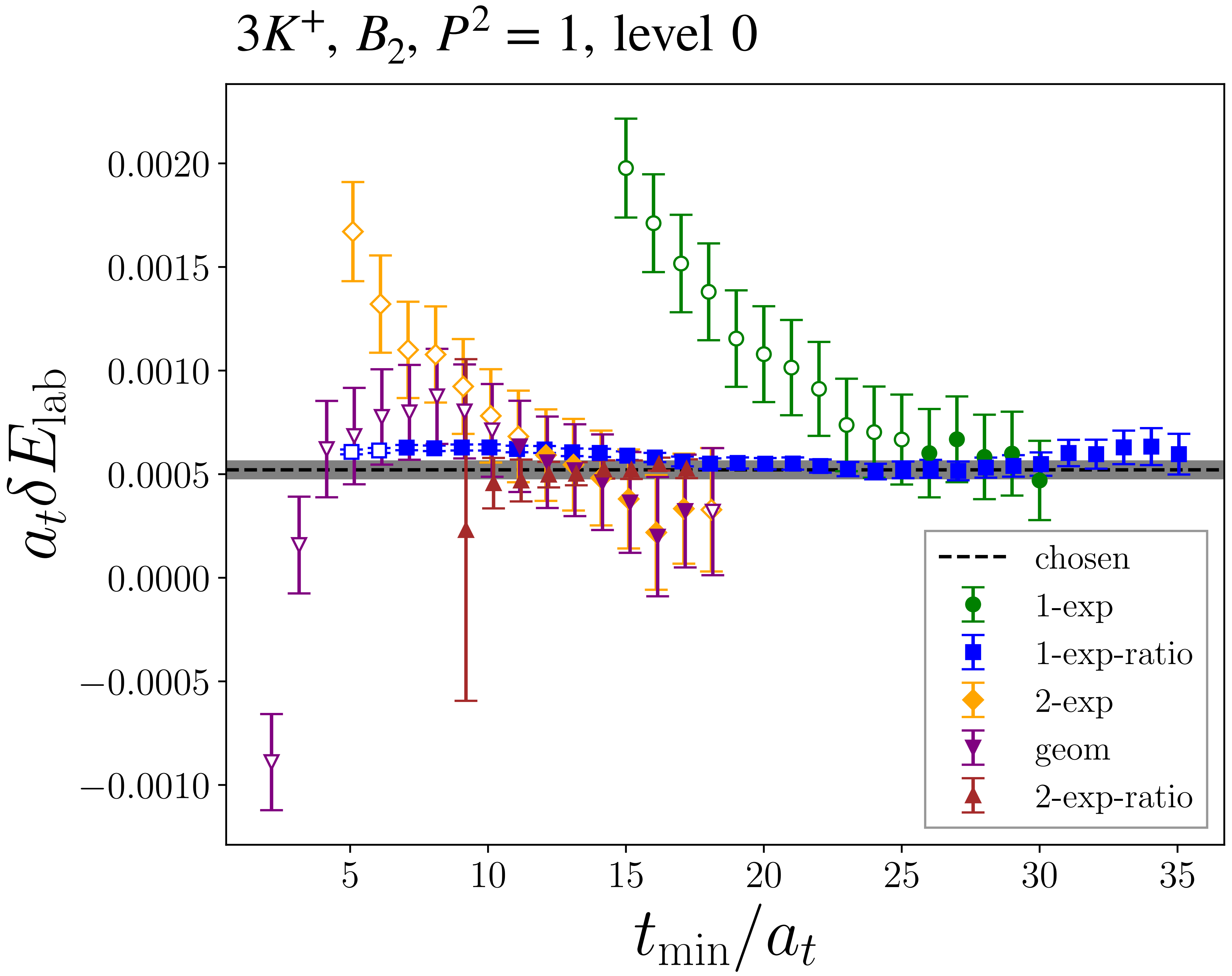


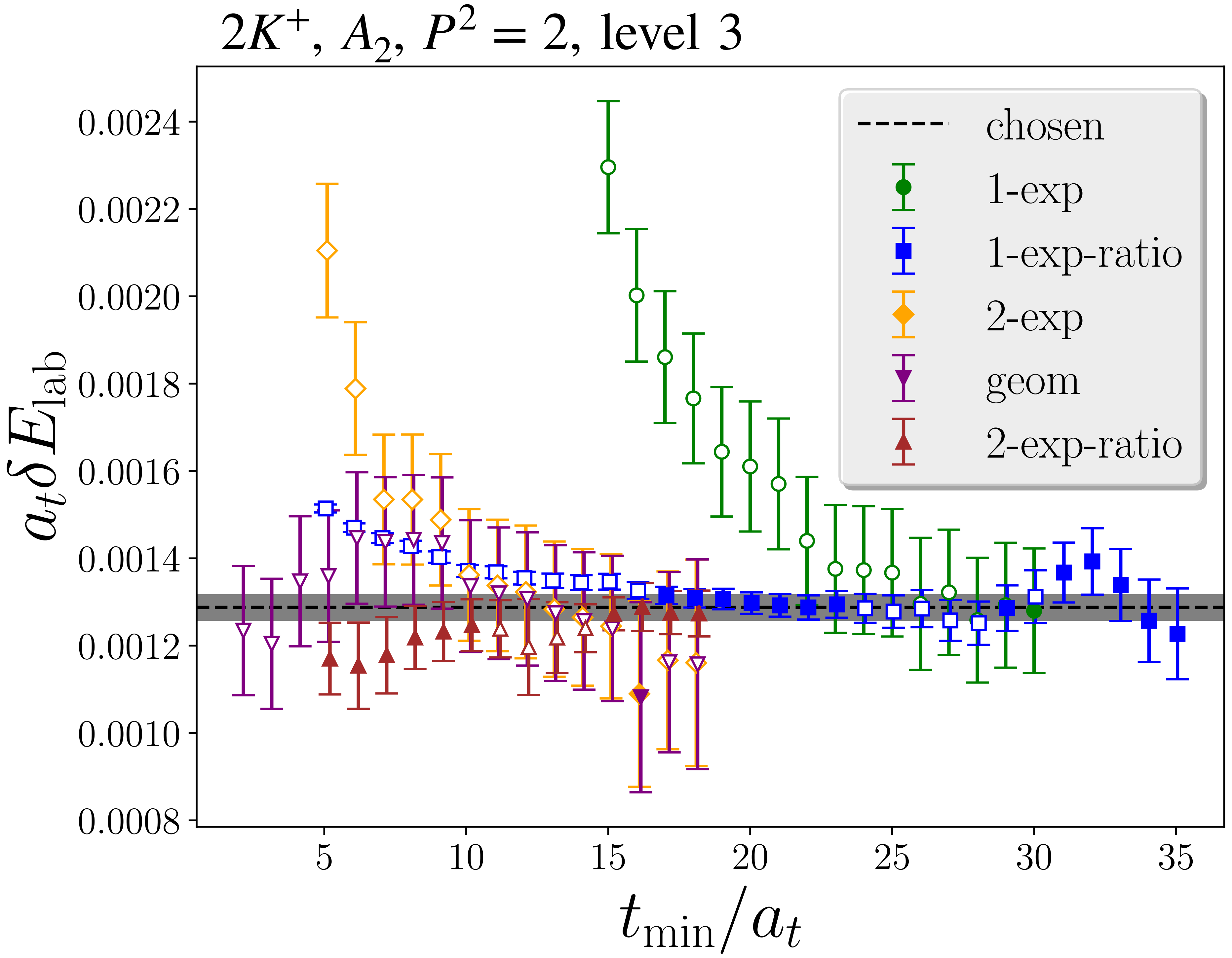
$$\mathcal{M}_3^{J=0}(E) = 9\left[\mathcal{M}_{3,00;00}^{(u,u)J=0}(E) + \frac{5}{4}\mathcal{M}_{3,22;22}^{(u,u)J=0}(E) - \frac{\sqrt{5}}{2}\left(\mathcal{M}_{3,22;00}^{(u,u)J=0}(E) + \mathcal{M}_{3,00;22}^{(u,u)J=0}(E)\right)\right],$$



Partial-wave projection
[Jackura, Briceño, 2312.00625]

t-min plots


$3K^+$, B_2, $P^2 = 1$, level 0
$a_t\delta E_{\rm lab}$
$t_{\rm min}/a_t$
0.0020
0.0015
0.0010
0.0005
0.0000
−0.0005
−0.0010
5
10
15
20
25
30
35
chosen
1-exp
1-exp-ratio
2-exp
geom
2-exp-ratio


$2K^+$, A_2, $P^2 = 2$, level 3
$a_t\delta E_{\rm lab}$
$t_{\rm min}/a_t$
0.0024
0.0022
0.0020
0.0018
0.0016
0.0014
0.0012
0.0010
0.0008
5
10
15
20
25
30
35
chosen
1-exp
1-exp-ratio
2-exp
geom
2-exp-ratio

Extracting energies

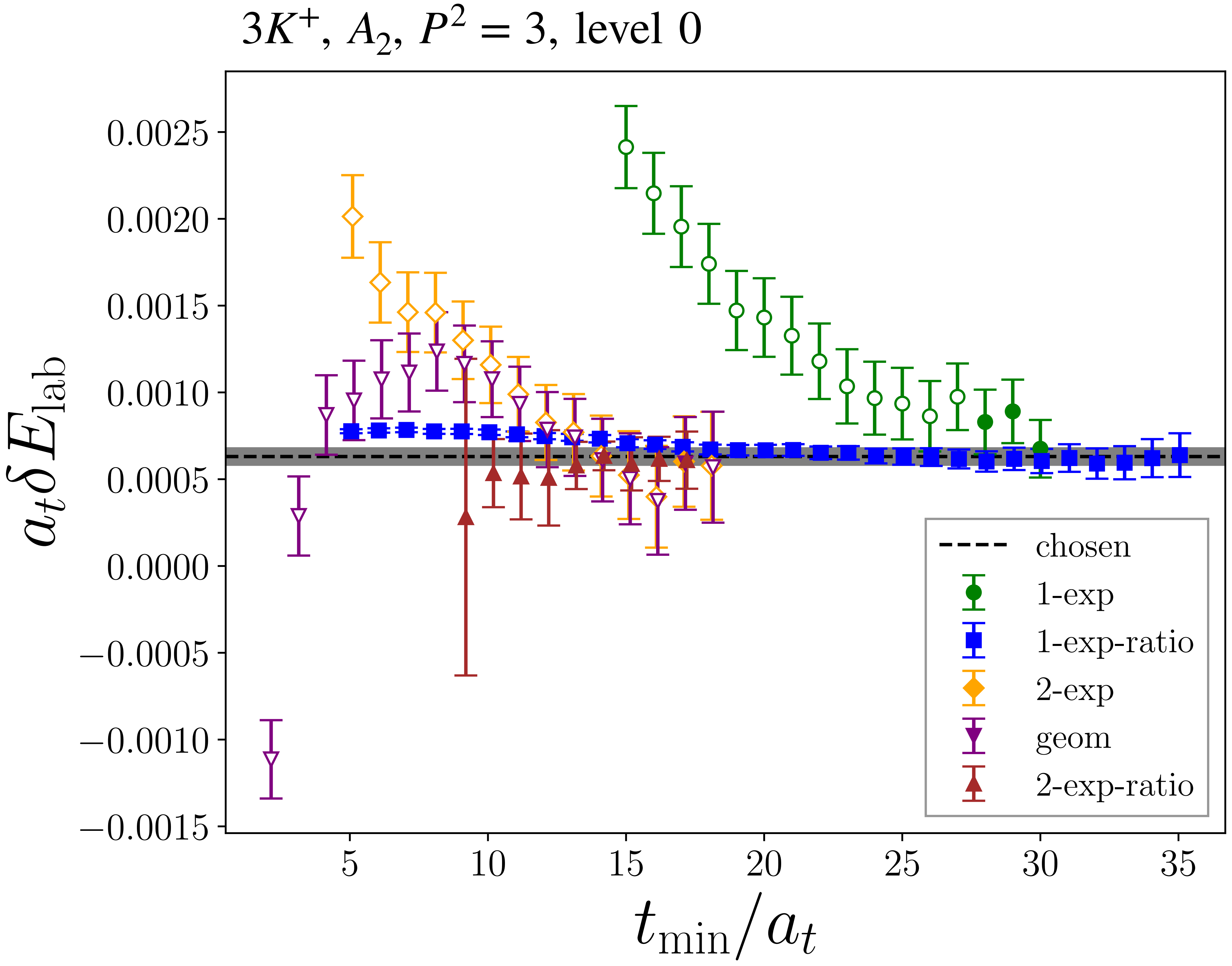
- Stochastic LapH method, multi-hadron operators

  [Morningstar et al, 1104.3870]

- GEVP and look for consistency between methods.
  - Single and double exponential
  - Ratio fit with single and double exponential

- Use ratio fit to benefit from correlated cancellations

$$R_n(t) = \frac{C_{\text{three-meson}}(t)}{C_{\text{meson}}(t) C_{\text{meson}}(t) C_{\text{meson}}(t)}$$



Fitting the spectrum

- Requires a correlated fit for several systems at one. For instance: $\pi\pi K + \pi\pi + \pi K$

- Fit energy shifts in the lab frame (“spectrum method”)

$$\chi^2(\vec{p}) = \sum_{ij} \left(\Delta E_{\mathrm{lab},i} - \Delta E^{\mathrm{QC}}_{\mathrm{lab},i}(\vec{p}) \right) \left(C^{-1} \right)_{ij} \left(\Delta E_{\mathrm{lab},j} - \Delta E^{\mathrm{QC}}_{\mathrm{lab},j}(\vec{p}) \right)$$

parameters in K-matrices

covariance matrix of lab-shifts

“predicted minus measured” lab-frame energy shifts

- Include several two-meson partial waves:
  - ◆ s and d waves for $\pi\pi$ and KK
  - ◆ s and p waves for πK

- Threshold expansion in three-body K matrix

$$\mathcal{K}_{\mathrm{df},3} = \mathcal{K}_0 + \mathcal{K}_1\Delta + \mathcal{K}_2\Delta^2 + \mathcal{K}_A\Delta_A + \mathcal{K}_B\Delta_B,$$

$$\boxed{\Delta \equiv \frac{s-9m^2}{9m^2}}$$

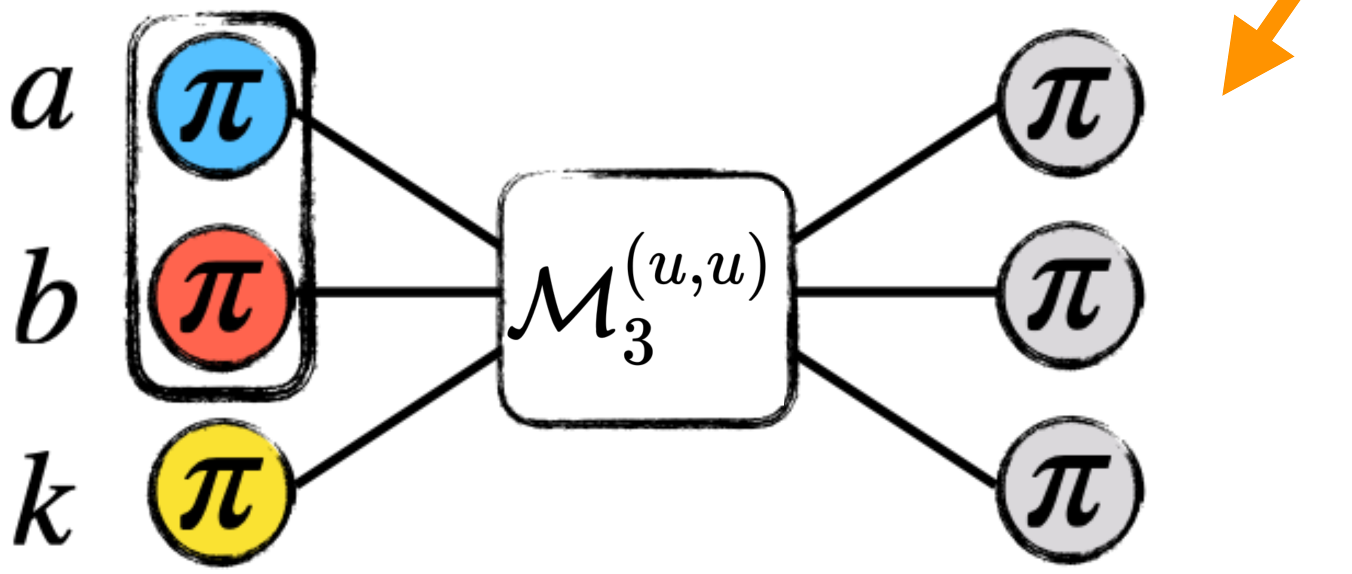
Functions of Mandelstam variables

Scattering amplitudes

- Physical amplitudes can be obtained after solving integral equations:

$$\mathcal{M}_3^{(u,u)}$$

"Unsymmetrized amplitude"



- Choice of spectator is fixed
- Needs "symmetrization"

$$\mathcal{M}_3 = \mathcal{S}\left[\mathcal{M}_3^{(u,u)}\right]$$

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Partial-wave projection
[Jackura, Briceño, 2312.00625]

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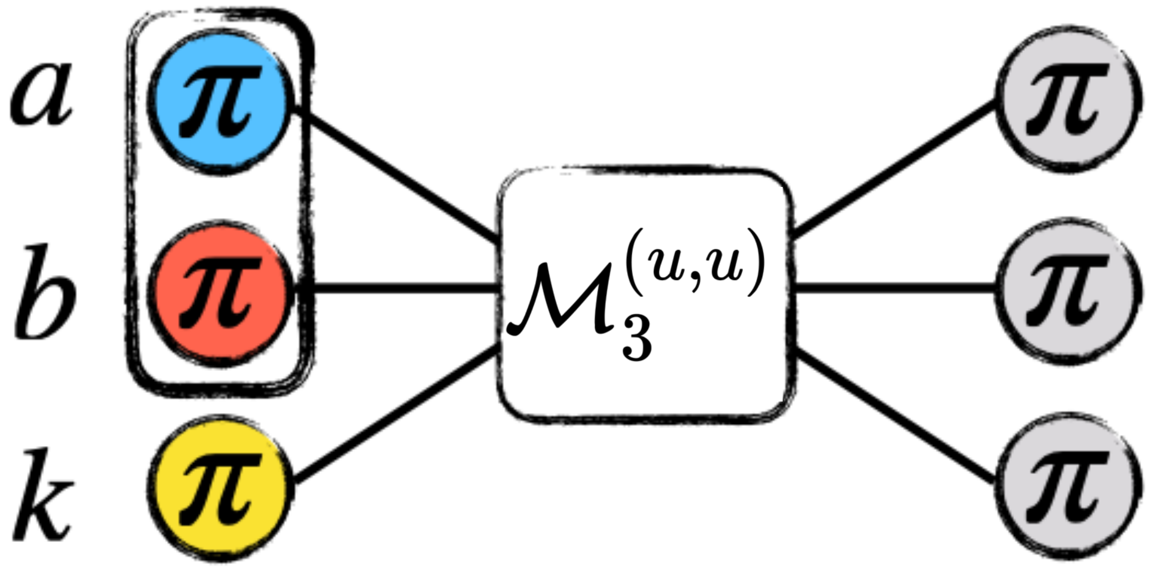
“Unsymmetrized amplitude”



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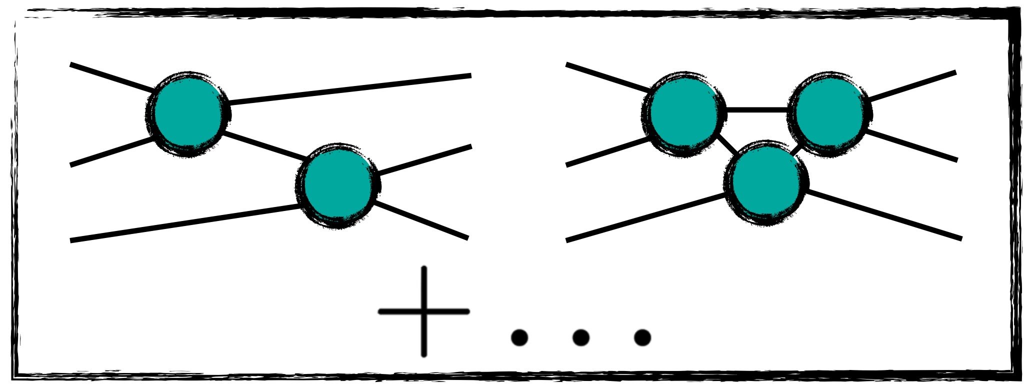
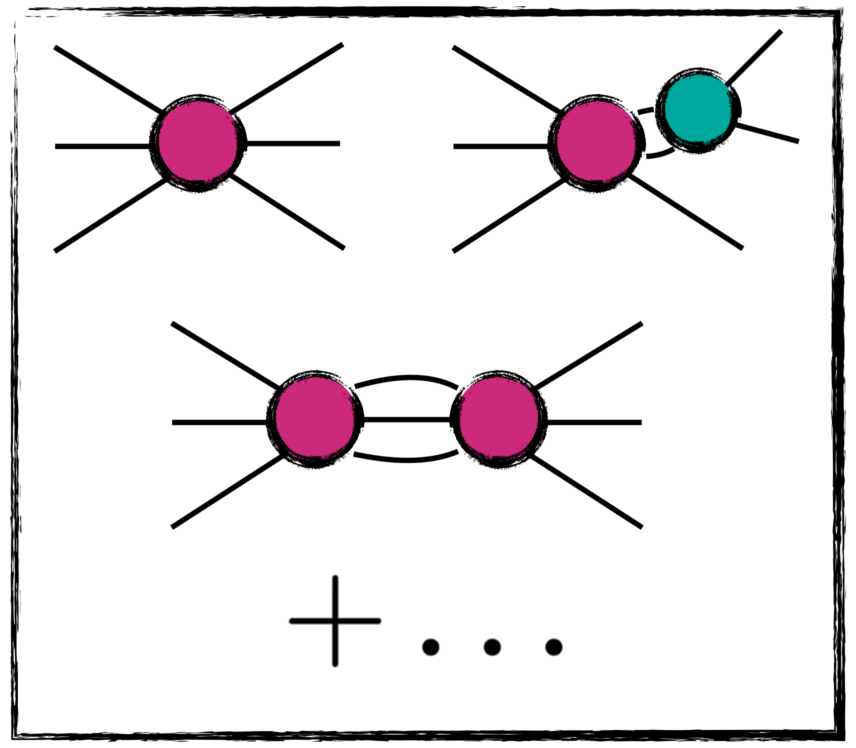
“ladder amplitude”



$$\mathcal{D}^{(u,u)} = -\mathcal{M}_2 G \mathcal{M}_2 - \int \mathcal{M}_2 G \mathcal{D}^{(u,u)}$$

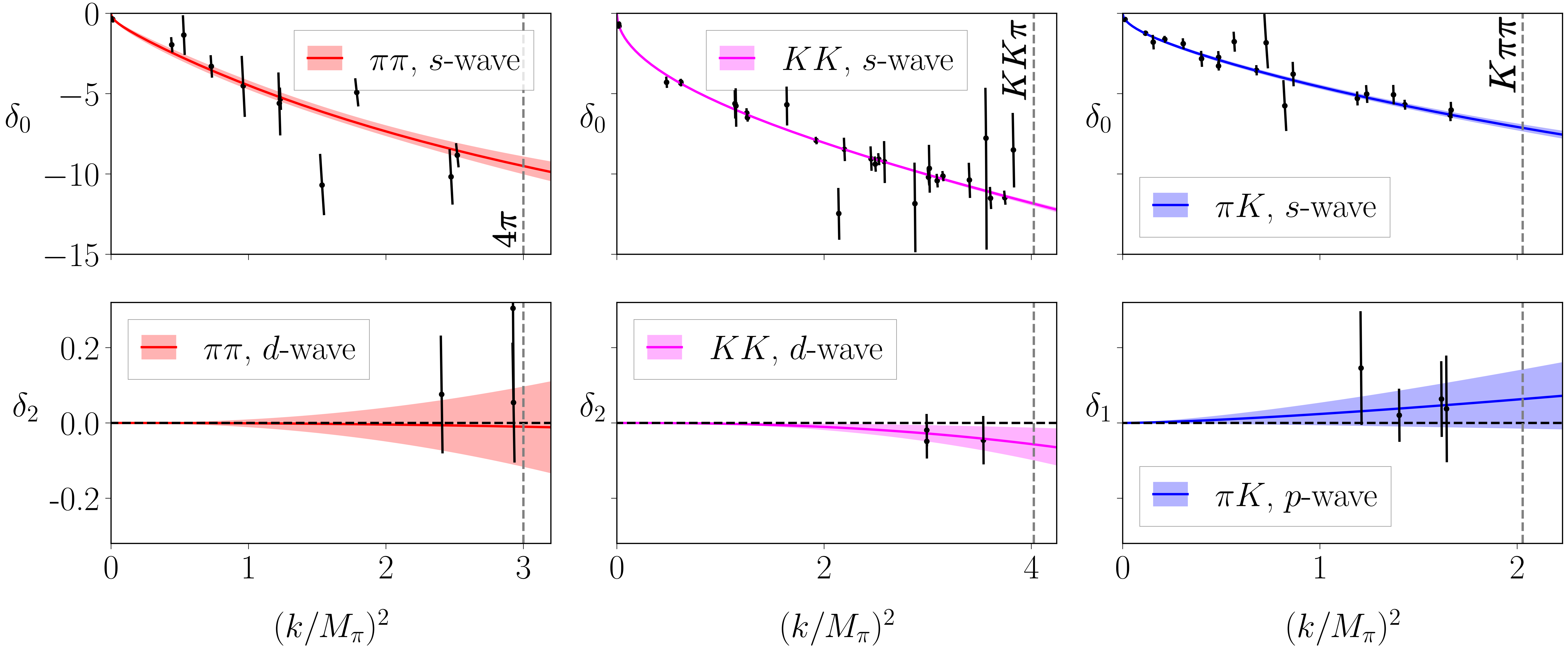
“divergence-free amplitude”



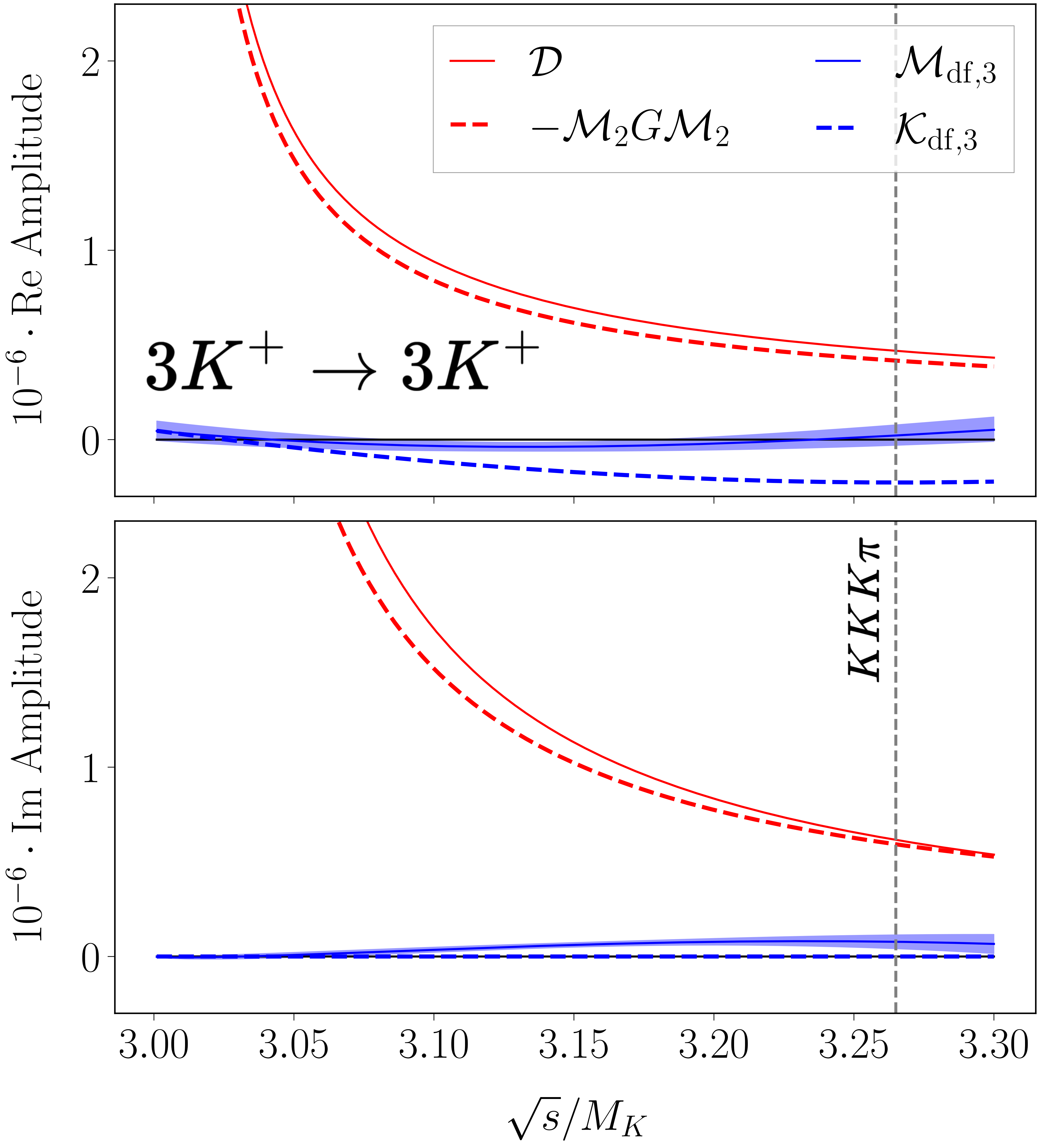
At least one three-body interaction

Two-body phase shift

- Required input for three-meson calculations

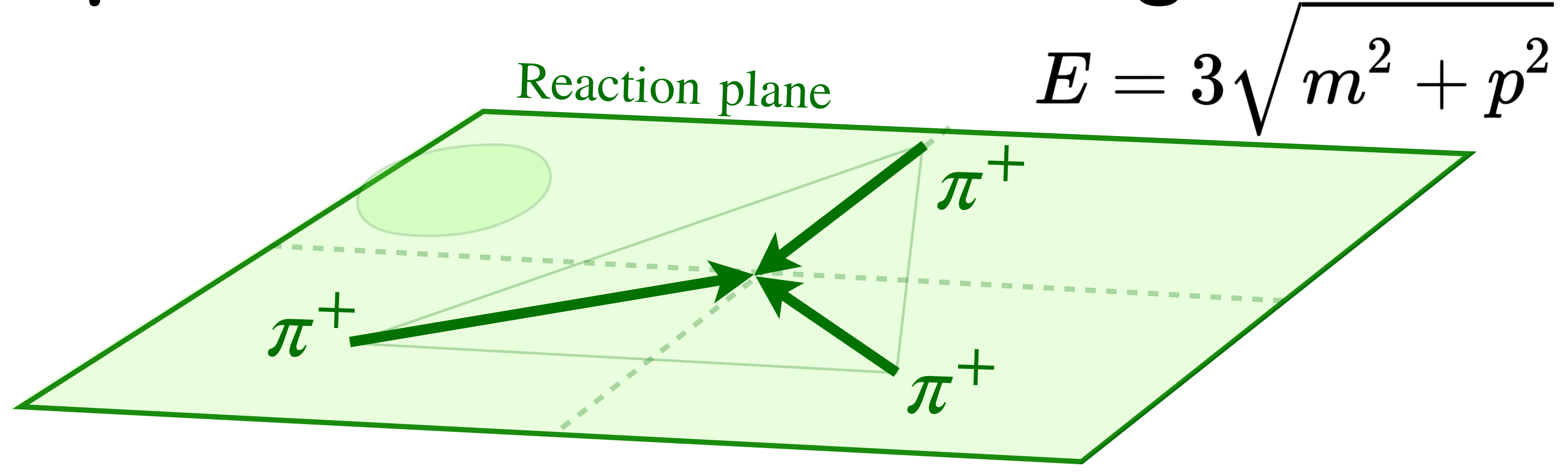


Three-kaon $J^P=0^-$ amplitude



$$\mathcal{M}_3 = \mathcal{D} + \mathcal{M}_{\mathrm{df},3}$$

Equilateral kinematic configuration



- **Partial-wave projected to $J^P = 0^-$**

[Jackura, Briceño, 2312.00625]