Review of lattice results on eta, eta'

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Introduction

Quarks cannot be observed directly but are bound in hadrons (at low energies):

- The lightest hadrons $\pi^\pm$, π^0, $K^\pm$, K^0, $\bar{K}^0$, η ("octet mesons") have masses from 135 MeV to 548 MeV.
- In addition there is a "flavor-singlet", the η'.
- For exact flavor symmetry ($m_u = m_d = m_s$) all 9 mesons should have the same mass.

However: $M_{\eta'} \approx 958\,\text{MeV} \gg M_{octet}$

Theoretical solution to this puzzle in QCD:

- Large mass of the η' is caused by the QCD vacuum structure and the $U(1)_A$ anomaly.
  Weinberg (1975), Belavin et al. (1975), t'Hooft (1976), Witten (1979), Veneziano (1979)
- The $U(1)$ axial current is anomalously broken, i.e. even for $m_q = 0$:
  Adler (1969), Jackiw and Bell (1969)

$$\partial_\mu A^0_\mu = \frac{N_f g^2}{32\pi^2} G^a_{\mu\nu} \tilde{G}^{a,\mu\nu} \neq 0$$

- Instantons with non-trivial topology provide non-perturbative explanation.
  Belavin et al. (1975) , t'Hooft (1976)
- The flavor-singlet η' remains massive as $m_l, m_s \to 0$.

Use lattice QCD to reproduce η' mass from first principles.

Introduction

For exact $SU(3)$ flavor symmetry one expects

- Flavor octet state $|\eta_8\rangle = \frac{1}{\sqrt{6}}(|\bar{u}u\rangle + |\bar{d}d\rangle - 2|\bar{s}s\rangle)$ (Pseudo-Goldstone boson)
- Flavor singlet state $|\eta_0\rangle = \frac{1}{\sqrt{3}}(|\bar{u}u\rangle + |\bar{d}d\rangle + |\bar{s}s\rangle)$ (related to $U(1)_A$ anomaly)

However, $SU(3)$ flavor symmetry is broken by large $m_s \gg m_u \approx m_d \equiv m_l$:

- Physical η, η' states are not flavor eigenstates but **mixtures**, e.g.

$$\begin{pmatrix} |\eta\rangle \\ |\eta'\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta_0 & -\sin\theta_8 \\ \sin\theta_0 & \cos\theta_8 \end{pmatrix} \begin{pmatrix} |\eta_0\rangle \\ |\eta_8\rangle \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} |\eta\rangle \\ |\eta'\rangle \end{pmatrix} = \begin{pmatrix} \cos\phi_l & -\sin\phi_s \\ \sin\phi_l & \cos\phi_s \end{pmatrix} \begin{pmatrix} |\eta_l\rangle \\ |\eta_s\rangle \end{pmatrix}$$

  in the **octet-singlet basis** or the **quark flavor basis** $|\eta_l\rangle = \frac{1}{\sqrt{2}}(|\bar{u}u\rangle + |\bar{d}d\rangle)$, $|\eta_s\rangle = |\bar{s}s\rangle$.

- Additional mixing possible in the physical world, e.g. with π^0 (for $m_u \neq m_d$), η_c, ...
- Mixing parameters related to further observables, e.g. $\Gamma_{\eta,\eta'\to\gamma\gamma}$, $\lim_{Q^2\to\infty} Q^2 F_{\eta,\eta'\to\gamma\gamma}(Q^2)$.
- Very recently: Direct LQCD calculations of $\eta,\eta' \to \gamma^*\gamma^*$ transition form factors (TFFs)

LQCD can be used to determine mixing parameters and TFFs.

Outline

1. How are η, η' simulated on the lattice?

   $\to$ Why are these calculations particularly challenging?

2. Overview on LQCD studies of η, η'.

3. Results for masses, mixing parameters, LECs and TFFs.

   $\to$ Focus on physical results with controlled systematics

4. Summary and outlook

η,η' on the lattice

Information on masses and mixing is encoded in (expectation values of) meson two-point correlation functions:

$$\mathcal{C}_{ij}(t) \sim \sum_{\mathbf{x}} \langle 0| O_i(x) O_j{}^{\dagger}(0) |0\rangle$$

- For η, η' use local pseudoscalar (or axialvector) interpolating operators $O_{i,j}$, e.g.:

$$\eta_l = \frac{1}{\sqrt{2}}(\bar{u}i\gamma_5 u + \bar{d}i\gamma_5 d)\,, \qquad \eta_s = \bar{s}i\gamma_5 s\,, \qquad \eta_c = \bar{c}i\gamma_5 c$$

$\rightarrow$ Choice of basis (quark flavor, octet-singlet) relevant for mixing.

- For e.g. $i = j$: $$\mathcal{C}_{ii}(t) = \sum_{n=\eta,\eta',\dots} \frac{|\langle 0|O_i|n\rangle|^2}{2M_n} \exp(-M_n t) \overset{t\gg 0}{\rightarrow} \frac{|\langle 0|O_i|\eta\rangle|^2}{2M_\eta} \exp(-M_\eta t)$$

$\rightarrow$ Ground state mass M_η can be extracted directly at sufficiently large t.

$\rightarrow$ Decay constants / mixing parameters related to physical amplitudes $A_i^n = \langle 0| O_i |n\rangle$.

- Higher states (η') from solving GEVP: $\mathcal{C}(t)v^{(n)}(t,t_0) = \lambda^{(n)}(t,t_0)\mathcal{C}(t_0)v^{(n)}(t,t_0)$

$\rightarrow$ Eigenvalues $\lambda^{(n)}(t,t_0)$ give mass of n-th state at $t \gg 0$.

$\rightarrow$ Eigenvectors $v^{(n)}(t,t_0)$ carry information on physical amplitudes $A_{l,s,\dots}^{\eta,\eta',\dots}$, $A_{8,0,\dots}^{\eta,\eta',\dots}$.

- Alternatively: Obtain masses and matrix elements from multi-state fits to $\mathcal{C}(t)$.

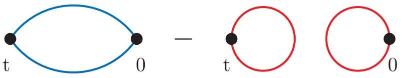
Quark disconnected diagrams

- Consider $O_i = O_j = \eta_l$:

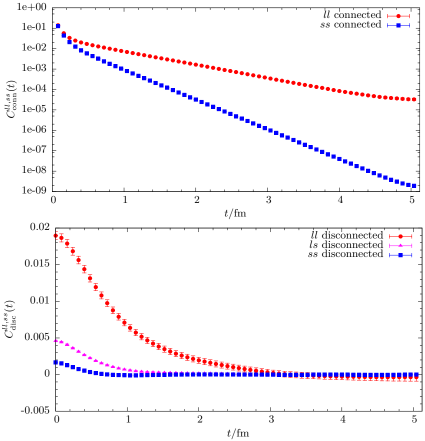
$$\mathcal{C}_{ll}(t) \sim \sum_{\mathbf{x}} \langle 0|\, \eta_l(x)\eta_l^\dagger(0)\,|0\rangle$$

$$\sim \mathrm{tr}\left[D_{0t}^{-1}\gamma_5 D_{t0}^{-1}\gamma_5\right] - \mathrm{tr}\left[D_{tt}^{-1}\gamma_5\right]\mathrm{tr}\left[D_{00}^{-1}\gamma_5\right]$$

- Quark-connected and -disconnected pieces:



- Lattice Dirac operator D_{xy} is a very large $(3\cdot 4\cdot L^3\cdot T)\times(3\cdot 4\cdot L^3\cdot T)$ – matrix
- Mixing mediated by quark-disconnected diagrams only, i.e. through $\mathcal{C}_{ls,sl}(t)$



Quark-connected and disconnected correlators;
tmWilson+Clover, $M_\pi = 139\,\mathrm{MeV}$, $a = 0.080\,\mathrm{fm}$

- Disconnected diagrams need **all-to-all** propagator $D_{xx}^{-1} \Rightarrow$ prohibitively expensive
- Use stochastic method instead (+ e.g. one-end trick)

Quark disconnected diagrams

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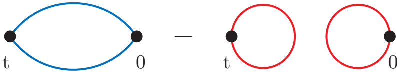

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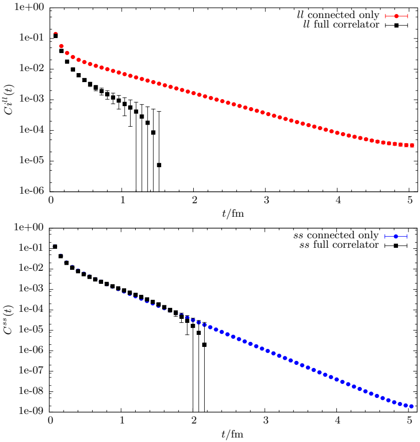

Quark-connected vs. full correlators; tmWilson+Clover, $M_\pi = 139\,\mathrm{MeV}$, $a = 0.080\,\mathrm{fm}$

→ **Very severe signal-to-noise problem; signal lost at $t \gtrsim 1\,\mathrm{fm}$ even for η**

→ **Computations always limited by gauge statistics; careful analysis required.**

Obtaining physical results

- **Fix bare parameters (a, m_l, m_s,...):**
  - Use known hadronic quantities (e.g. M_π^{phys}, M_K^{phys}, ...) $\rightarrow$ Further observables are predictions.
- **Control discretization effects:**
  - Simulate at different (small) values of a.
  - **Perform continuum extrapolation.**
  - With modern LQCD calculations lattice artifacts are typically $\propto a^2$.
- **Correct for unphysical quark masses:**
  - Simulate at several light and strange quark masses, or tune $m_s = m_s^{\mathrm{phys}}$
  - **Perform chiral extrapolation.**
  - State-of-the-art lattice simulations include physical quark masses.
  - Can determine LECs from fits to unphysical masses.
- **Control finite volume effects:**
  - Simulate several physical volumes.
  - Perform infinite volume extrapolation / make sure that FS effects are negligible $M_\pi L \gtrsim 4$.

Overview of $N_f = 2$ studies

- Mostly older studies; spread over 2-3 decades, often unclear systematics:
  - $\to$ typically $\mathcal{O}(1)$ ensembles
  - $\to$ no continuum and / or chiral limit
  - $\to$ no scale setting available ...
- No direct correpondence to the physical world; no mixing, only a single "η_0"
- Overall agreement for M_{η_0} good; little M_π-dependence.
- m_l^{phys} reached in 2019 by ETMC.

Confirmed $M_{\eta_0} \neq 0$ in chiral limit.

- Further applications: tests of $N_f = 2$ Veneziano-Witten formula, glueball mixing, $\eta_c \to \gamma\gamma$, $J/\psi \to \eta_0\gamma$ ...

e.g. **Dimpopoulos et al.**, PRD 99, 034511 (2019), **Liu et al.**, PRD 102, 034502 (2020), **Jiang et al.**, PRD 107, 094510 (2023) **Jiang et al.**, PRL 130, 061901 (2023)

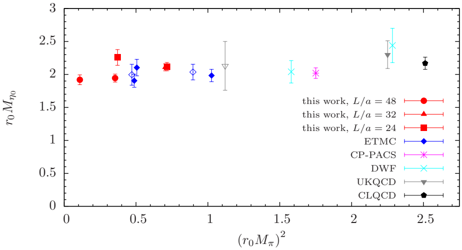

Figure reproduced from PRD 99, 034511 (2019). $N_f = 2$ data from:

ETMC:	PRD 99, 034511 (2019)
ETMC:	Eur. Phys. J. C 58, 261 (2008)
CP-PACS:	PRD 67, 074503 (2003)
DWF:	Prog. Theor. Phys. 119, 599 (2008)
UKQCD:	PRD 70, 014501 (2004)
CLQCD:	Chin. Phys. C 42, 093103 (2018)

(even older studies exist)

$\to$ $N_f = 2$ **flavor singlet studies should be considered a closed chapter.**

Overview of $N_f = 2+1$ and $N_f = 2+1+1$ studies

Lattice calculations including dynamical strange quarks exist since $\lesssim 15$ years:

- Several older studies (e.g. single / few ensembles, no phys. extrapolation, large uncertainties ...)
  Dudek et al., PRD 83, 111502 (2011) **Gregory et al.**, PRD 86, 014504 (2012) **Ottnad et al.**, JHEP 11 (2012) 048 **Ottnad et al.**, NPB 896 470-492 (2015) **Bali et al.**, PRD 91, 014503 (2015) **Fukaya et al.**, PRD 92, 111501 (2015)

- UKQCD 2010: Early attempt of a chiral extrapolation on three DWF ensembles. **Christ et al.**, PRL 105, 241601 (2010)

- ETMC 2013-2023: First physical results for $M_{\eta,\eta'}$, ϕ and $f_{l,s}$, VW formula, $\eta \to \gamma\gamma$ TFF
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  $\to$ Update including several ensembles at physical quark mass: work in progress

- RQCD 2021: Physical results for masses + mixing parameters on CLS ensembles **Bali et al.**, JHEP 08 (2021) 137
  $\to$ Axialvector + gluonic matrix elements, scale dependence, determination of NLO U(3) χPT LECs

- BMW 2023/????: Physical results for $\eta,\eta' \to \gamma\gamma$ TFFs on staggered ensembles **Gerardin et al.**, arXiv:2305.04570
  $\to$ Planned publication on masses / mixing (not yet available); cf. remark in arXiv:2305.04570

- CSSM/QCDSF/UKQCD 2021: First, very exploratory QCD+QED study **Kordov et al.**, PRD 104, 114514 (2021)

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$\to$ Remaining talk: **Studies with physical results.**

ETMC '18: Setup and masses

Ottnad et al., PRD 97, 054508 (2018)

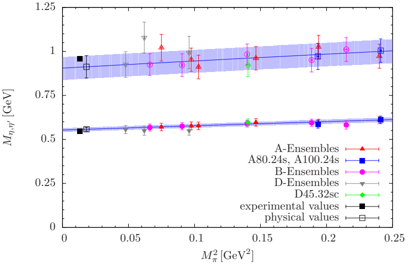
- 17 ensembles with $N_f = 2+1+1$ flavors of Wilson twisted-mass quarks generated by ETMC.
- $M_\pi \in [220, \ldots, 490]$ MeV, $a \in [0.061, 0.081, 0.089]$ fm.
- Automatic $\mathcal{O}(a)$ improvement.
- Ansatz for phys. extrapolations ($P = \eta, \eta'$):

$$(r_0 M_P)^2 = (r_0 \mathring{M}_P)^2 + \sum_{i=\pi,K} c_i (r_0 M_i)^2 + c_\beta \left(\frac{a}{r_0}\right)^2 .$$

- Scale setting: Sommer parameter $r_0 = 0.474(14)\,\mathrm{fm}$.
  Carrasco et al., NPB 887 (2014) 19-68
- **Results in good agreement with experiment.**
- Results for $M_{\eta,\eta'}$ also agree with 2013 analysis.
  Michael et al. PRL 111, 181602 (2013)
- $M_{\eta,\eta'}$ used as input for VW formula test:

$$\chi_\infty^{\mathrm{YM}} = (185.3(5.6)_{\mathrm{stat+sys}}\,\mathrm{MeV})^4 \quad \text{vs.} \quad \chi_\infty^{\mathrm{dyn}} = (182.6(8.3)_{\mathrm{stat+sys}}\,\mathrm{MeV})^4$$

Cichy et al., JHEP 09 (2015) 020, **Ottnad**, PoS CD2018 (2019) 077



Physical results:

$$M_\eta = 557(11)_{\mathrm{stat}}(03)_{\chi PT}\,\mathrm{MeV}$$
$$M_{\eta'} = 911(64)_{\mathrm{stat}}(03)_{\chi PT}\,\mathrm{MeV}$$

ETMC '18: Mixing

Ottnad et al., PRD 97, 054508 (2018)

- Computed pseudoscalar matrix elements

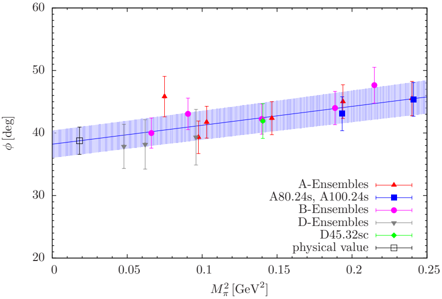
$$H^i_{\mathrm{P}} = 2m_i < 0|P^i|\mathrm{P} >, \quad \mathrm{P} = \eta, \eta',$$

- Related to axial vector matrix elements through χPT (with $1/N_c$ counting):

$$\begin{pmatrix} H^l_\eta & H^s_\eta \\ H^l_{\eta'} & H^s_{\eta'} \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \mathrm{diag}\left(f_l M_\pi^2, f_s \left(2M_K^2 - M_\pi^2\right)\right).$$

- Scale dependence neglected in Feldmann-Kroll-Stech (FKS) scheme.
  Feldmann et al., PRD 58, 114006 (1998), PLB 449 (1999) 339-346

- Splitting $|\phi_l - \phi_s|$ expected to be small: Only $1/N_c$ corrections, no SU(3) flavor breaking terms at NLO.



Physical result: $\phi_{\mathrm{phys}} = 38.8^\circ(2.2)_{\mathrm{stat}}(2.4)_{\chi PT}$

Confirmed by lattice data, i.e. $|\phi_l - \phi_s| = 2.8(1.1)_{\mathrm{stat}}(2.6)^\circ_{\chi PT}$

Excellent agreement with pheno results, e.g. $\phi^{\mathrm{pheno}} = 39.3^\circ(1.0)$. Feldmann, Int. J. Mod. Phys. A15 (2000) 159-207

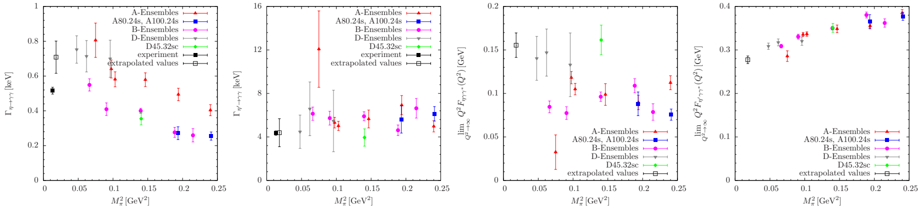
First ever lattice results for η,η' decay constant parameters:

$$(f_l/f_\pi)_{\mathrm{phys}} = 0.960(37)_{\mathrm{stat}}(46)_{\chi PT} \quad \to \quad f_{l,\mathrm{phys}} = 125(5)_{\mathrm{stat}}(6)_{\chi PT}\,\mathrm{MeV}$$

$$(f_s/f_K)_{\mathrm{phys}} = 1.143(23)_{\mathrm{stat}}(04)_{\chi PT} \quad \to \quad f_{s,\mathrm{phys}} = 178(4)_{\mathrm{stat}}(1)_{\chi PT}\,\mathrm{MeV}$$

ETMC '18: $\Gamma_{\eta,\eta'\to\gamma\gamma}$ and TFFs at large-Q^2

Ottnad et al., PRD 97, 054508 (2018)



χPT relates ϕ, f_l, f_s to decay widths and large-Q^2 behavior of TFFs:

$$\Gamma_{\eta\to\gamma\gamma} = \frac{\alpha_{\mathrm{QED}}^2 M_\eta^3}{288\pi^3} \cdot \left[\frac{5\cos\phi}{f_l} - \frac{\sqrt{2}\sin\phi}{f_s}\right]^2 , \qquad \hat{F}_{\eta\gamma\gamma^*} \equiv \lim_{Q^2\to\infty} Q^2 F_{\eta\gamma\gamma^*}(Q^2) = \frac{\sqrt{2}}{3} \cdot \left[5 f_l \cos\phi - \sqrt{2} f_s \sin\phi\right] ,$$

$$\Gamma_{\eta'\to\gamma\gamma} = \frac{\alpha_{\mathrm{QED}}^2 M_{\eta'}^3}{288\pi^3} \cdot \left[\frac{5\sin\phi}{f_l} + \frac{\sqrt{2}\cos\phi}{f_s}\right]^2 , \qquad \hat{F}_{\eta\gamma\gamma^*} \equiv \lim_{Q^2\to\infty} Q^2 F_{\eta'\gamma\gamma^*}(Q^2) = \frac{\sqrt{2}}{3} \cdot \left[5 f_l \sin\phi + \sqrt{2} f_s \cos\phi\right] .$$

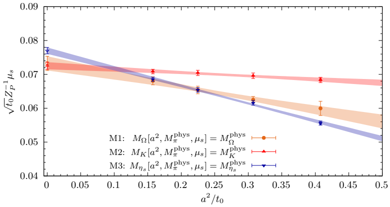
$$\Gamma_{\eta\to\gamma\gamma} = 0.71(9)_{\mathrm{stat}}(7)_{\chi PT}\ \mathrm{keV}, \qquad \hat{F}_{\eta\gamma\gamma^*} = 155(14)_{\mathrm{stat}}(23)_{\chi PT}\ \mathrm{MeV},$$
$$\Gamma_{\eta'\to\gamma\gamma} = 4.4(1.3)_{\mathrm{stat}}(0.6)_{\chi PT}\ \mathrm{keV}, \qquad \hat{F}_{\eta'\gamma\gamma^*} = 277(09)_{\mathrm{stat}}(01)_{\chi PT}\ \mathrm{MeV}.$$

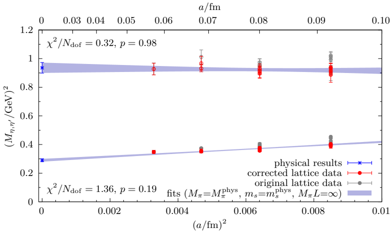
- Decay widths in reasonably good agreement with PDG values.
- However: Scale dependence neglected; potential issue for $\hat{F}_{\eta'\gamma\gamma^*}$, $\hat{F}_{\eta\gamma\gamma^*}$

ETMC '24 (preliminary): Setup

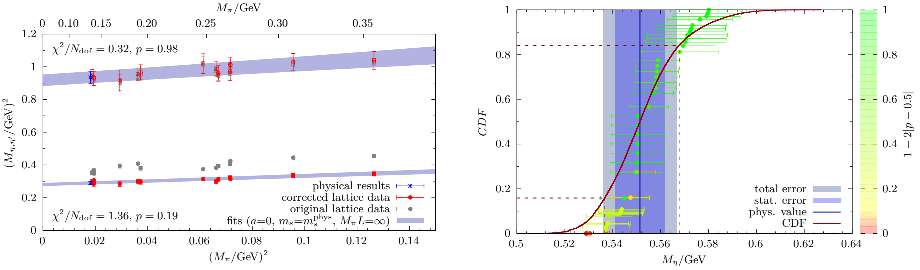
- 15 ensembles with $N_f = 2+1+1$ twisted mass Wilson+Clover quarks.
- $M_\pi \in [138, \ldots, 350]\,\mathrm{MeV}$, physical m_s^{sea}
- **3 ensembles at $m_{l,s}^{\mathrm{phys}}$, 6 with $M_\pi < 200\,\mathrm{MeV}$.**
- Fits dominated by ensembles at $m_{l,s}^{\mathrm{phys}}$
- Four values of $a \in [0.057, \ldots, 0.092]\,\mathrm{fm}$.
- Osterwalder-Seiler discretization for valence strange quarks. Osterwalder et al., Annals Phys. 110 (1978) 440
- Use three choices for μ_s^{val}-matching $\to$ different approach to the continuum
- **Fully controlled systematics / error budget; separation of stat and sys. errors from model averaging.**
- $\mathcal{O}(100)$ model variations per observable

$$\{\mu_s\text{-matchings}\} \bigotimes \{\text{CCF models}\} \bigotimes \{\text{data cuts}\}$$





ETMC '24 (preliminary): Masses



Physical results: $M_\eta = 551(10)_{\rm stat}(12)_{\rm sys}[16]_{\rm total}\,\text{MeV}$, $M_{\eta'} = 971(19)_{\rm stat}(06)_{\rm sys}[20]_{\rm total}\,\text{MeV}$

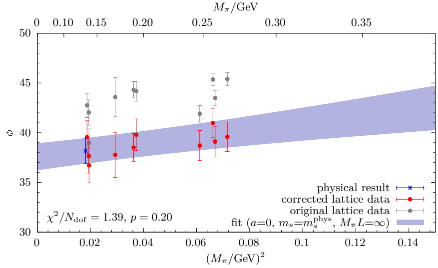
- Excellent agreement with experiment ($M_\eta^{\rm exp} = 547.862(17)\,\text{MeV}$, $M_{\eta'}^{\rm exp} = 957.78(6)\,\text{MeV}$).
- Error on M'_η improved by factor ~ 3 compared to our previous results

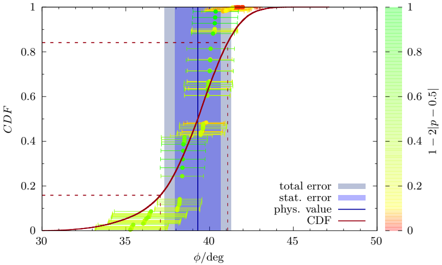
$$M_\eta = 557(11)_{\rm stat}(03)_{\chi PT}\ \text{MeV}, \quad M_{\eta'} = 911(64)_{\rm stat}(03)_{\chi PT}\ \text{MeV}$$

- Improved control over systematic effects (chiral + continuum + FS).
- Scale setting: $\sqrt{t_0^{\rm phys}} = 0.14436(61)\,\text{fm}$.

Alexandrou et al., PRD 104 (2021) 7, 074520

ETMC '24 (preliminary): Mixing





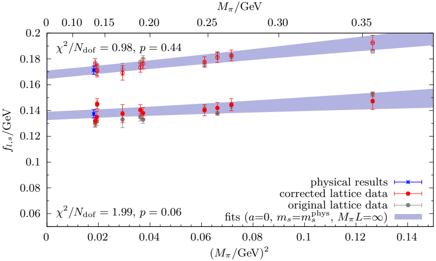
Physical result: $\phi = 39.6(1.4)_{\text{stat}}(1.5)_{\text{sys}}[2.0]^\circ_{\text{total}}$

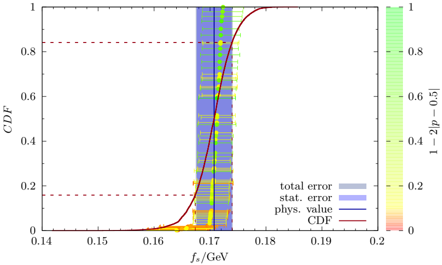
- $\Delta\phi$ improved by factor ~ 1.5 compared to old result $\phi = 38.8(2.2)_{\text{stat}}(2.4)^\circ_{\chi PT}$.
- Value for ϕ in excellent agreement with pheno determinations, e.g.

	ϕ_l	ϕ_s	
R. Escribano et al. (2016)	$39.6(2.3)^\circ$	$40.8(1.8)^\circ$	PRD 94 (2016), 054033
R. Escribano et al. (2015)	$39.3(1.2)^\circ$	$39.2(1.2)^\circ$	EPJC 75, 414 (2015)
Th. Feldmann (2000)	$39.3(1.0)^\circ$	$39.3(1.0)^\circ$	Int. J. Mod. Phys. A 15 (2000)

- Compatible with RQCD $\phi_l(\mu{=}2\,\text{GeV}) = 36.2\,\binom{1.1}{2.0}_{\text{stat}}\binom{1.3}{0.4}^\circ_{\text{sys}}$ and $\phi_s(\mu{=}2\,\text{GeV}) = 37.9\,\binom{1.0}{1.3}_{\text{stat}}\binom{1.4}{0.8}^\circ_{\text{sys}}$, although scale dependence is neglected in FKS scheme. Bali et al., JHEP 08 (2021) 137

ETMC '24 (preliminary): Decay constant parameters





Physical results: $f_l = 138.3(4.0)_{\rm stat}(1.8)_{\rm sys}[4.4]_{\rm total}\,\text{MeV}$, $f_s = 170.7(3.2)_{\rm stat}(1.2)_{\rm sys}[3.3]_{\rm total}\,\text{MeV}$

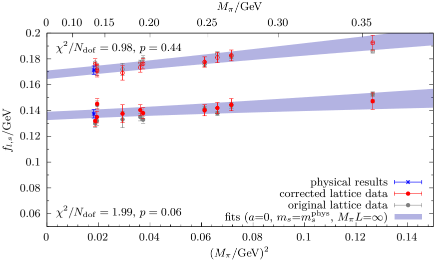
- f_l increased, f_s decreased compared to 2018 analysis, i.e.

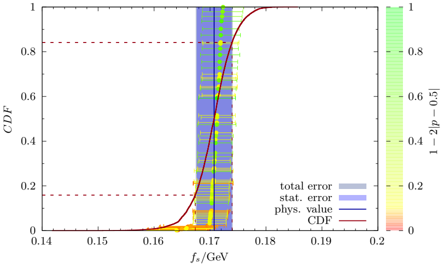
$$f_l = 125(5)_{\rm stat}(6)_{\chi PT}\,\text{MeV}, \quad f_s = 178(4)_{\rm stat}(1)_{\chi PT}\,\text{MeV}$$

- Improved control over systematic effects of physical extrapolations; particularly for f_l.
- Physical extrapolation of ratios: $f_l/f_\pi = 1.057(28)_{\rm stat}(27)_{\rm sys}$ and $f_s/f_K = 1.105(20)_{\rm stat}(13)_{\rm sys}$

$$\Rightarrow \quad f_l = 137.6(3.6)_{\rm stat}(3.5)_{\rm sys}\,\text{MeV}, \quad f_s = 172.0(3.1)_{\rm stat}(2.3)_{\rm sys}\,\text{MeV}$$

ETMC '24 (preliminary): Decay constant parameters





Physical results: $f_l = 138.3(4.0)_{\text{stat}}(1.8)_{\text{sys}}[4.4]_{\text{total}}\,\text{MeV}, \quad f_s = 170.7(3.2)_{\text{stat}}(1.2)_{\text{sys}}[3.3]_{\text{total}}\,\text{MeV}$

- Errors on f_s quite competitive; new analysis in better agreement with pheno results for f_l, e.g.

	f_l	f_s	
R. Escribano et al. (2016)	$134.2(5.2)\,\text{MeV}$	$177.2(5.2)\,\text{MeV}$	PRD 94 (2016), 054033
R. Escribano et al. (2015)	$139.6(12.7)\,\text{MeV}$	$181.0(18.3)\,\text{MeV}$	EPJC 75, 414 (2015)
Th. Feldmann (2000)	$139.3(2.5)\,\text{MeV}$	$174.5(7.8)\,\text{MeV}$	Int. J. Mod. Phys. A 15 (2000)

- RQCD results at $\mu = 2\,\text{GeV}$ shows tension for f_l, but good agreement for f_s:

$$f_l = 124.9\,\left({}^{1.7}_{2.9}\right)_{\text{stat}}\left({}^{4.2}_{2.5}\right)_{\text{sys}}(1.6)_{t_0}\,\text{MeV}, \quad f_s = 175.8\,\left({}^{2.4}_{2.3}\right)_{\text{stat}}\left({}^{3.8}_{6.1}\right)_{\text{sys}}(2.3)_{t_0}\,\text{MeV}$$

Bali et al., JHEP 08 (2021) 137

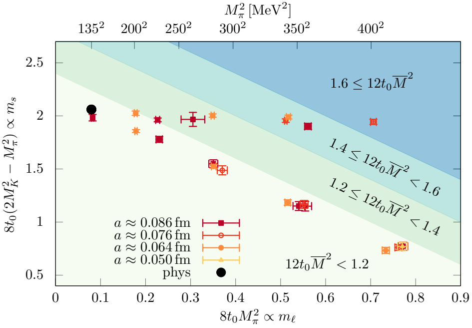
RQCD '21: Setup

Bali et al., JHEP 08 (2021) 137

- 21 ensembles with $N_f = 2+1$ Wilson-Clover quarks generated by CLS.
- Two quark mass trajectories, i.e. $\mathrm{tr}[M] = \mathrm{const}$ and $m_s \approx m_s^{\mathrm{phys}}$
- $M_\pi \in [135\ldots420]\,\mathrm{MeV}$
- Four lattice spacings $a \in [0.050\ldots0.086]\,\mathrm{fm}$
- Computation of axialvector matrix elements:

  $\langle 0|\, A^i_\mu \,|\mathrm{P}\rangle = iF^i_P p_\mu\,, \quad \mathrm{P} = \eta,\eta'\,, \quad i = 0,8$

  $\to$ **direct extraction of $F_{0,8}$ $\theta_{0,8}$ ($f_{l,s}$, $\phi_{l,s}$)**



- Phys. extrapolation using NLO large-N_c χPT. $\to$ determination of LECs, i.e. F, M_0, $L_{5,8}$, $\Lambda_{1,2}$
- Study of scale dependence.
- Gluonic matrix elements $a_P(\mu) = \langle 0|2\omega|P\rangle$, $P = \eta,\eta'$ from singlet axial Ward identity (AWI).

RQCD '21: Physical extrapolation

Bali et al., JHEP 08 (2021) 137

Physical extrapolation using NLO large-N_c χPT

$$(\mu_8^{\rm NLO})^2 = (\mu_8^{\rm LO})^2 + \frac{8}{3F^2}(2L_8 - L_5)\,\delta M^4,$$

$$(\mu_0^{\rm NLO})^2 = (\mu_0^{\rm LO})^2 + \frac{4}{3F^2}(2L_8 - L_5)\,\delta M^4 - \frac{8}{F^2}L_5\overline{M}^2 M_0^2 - \tilde\Lambda\overline{M}^2 - \Lambda_1 M_0^2,$$

$$(\mu_{80}^{\rm NLO})^2 = (\mu_{80}^{\rm LO})^2 - \frac{4\sqrt{2}}{3F^2}(2L_8 - L_5)\,\delta M^4 + \frac{4\sqrt{2}}{3F^2}L_5 M_0^2\delta M^2 + \frac{\sqrt{2}}{6}\tilde\Lambda\delta M^2,$$

$$F_\eta^8 = F\left[\cos\theta + \frac{4L_5}{3F^2}\left(3\cos\theta\overline{M}^2 + (\sqrt{2}\sin\theta + \cos\theta)\delta M^2\right)\right],$$

$$F_{\eta'}^8 = F\left[\sin\theta + \frac{4L_5}{3F^2}\left(3\sin\theta\overline{M}^2 + (\sin\theta - \sqrt{2}\cos\theta)\delta M^2\right)\right],$$

$$F_\eta^0 = -F\left[\sin\theta\left(1 + \frac{\Lambda_1}{2}\right) + \frac{4L_5}{3F^2}\left(3\sin\theta\overline{M}^2 + \sqrt{2}\cos\theta\delta M^2\right)\right],$$

$$F_{\eta'}^0 = F\left[\cos\theta\left(1 + \frac{\Lambda_1}{2}\right) + \frac{4L_5}{3F^2}\left(3\cos\theta\overline{M}^2 - \sqrt{2}\sin\theta\delta M^2\right)\right],$$

LO expressions:

$$(\mu_8^{\rm LO})^2 = \overline{M}^2 + \frac{1}{3}\delta M^2,$$

$$(\mu_0^{\rm LO})^2 = \overline{M}^2 + M_0^2,$$

$$(\mu_{80}^{\rm LO})^2 = -\frac{\sqrt{2}}{3}\delta M^2,$$

$$\theta = \frac{1}{2}\arctan\left(\frac{-2\sqrt{2}\delta M^2}{3M_0^2 - \delta M^2}\right),$$

where

$$\overline{M}^2 = \frac{2}{3}(2m_l + m_s),$$

$$\delta M^2 = 2B_0(m_s - m_l),$$

$$M_0^2 = M_0^2(\mu) = \frac{2N_f}{F_\pi^2}\chi_t.$$

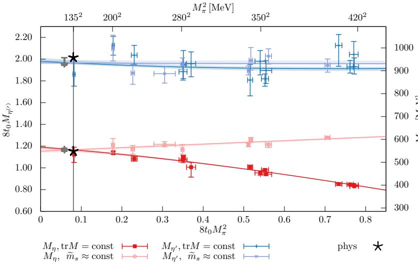
$\tilde\Lambda = \Lambda_1(\mu) - 2\Lambda_2(\mu)$ is scale independent.

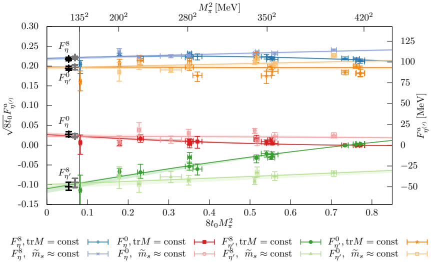
L_i are different from $SU(3)$ LECs!

- Masses and matrix elements share same LECs F, M_0 (LO), L_5, L_8, $\Lambda_1(\mu)$, $\Lambda_2(\mu)$ (NLO)
- **Simultanous fits** for masses $M_{\eta,\eta'}$ and matrix elements $F^{0,8}_{\eta,\eta'}$.
- Complemented by terms for lattice artifacts

RQCD '21: Masses and mixing parameters

Bali et al., JHEP 08 (2021) 137





Physical results:

$$M_\eta = 554.7\,(^{4.0}_{6.6})_{\rm stat}\,(^{2.4}_{2.7})_{\rm sys}\,(7.0)_{t_0}\,[9.2]_{\rm total}\,\mathrm{MeV}$$

$$M_{\eta'} = 929.9\,(^{12.9}_{6.0})_{\rm stat}\,(^{22.9}_{3.3})_{\rm sys}\,(11.7)_{t_0}\,[21]_{\rm total}\,\mathrm{MeV}$$

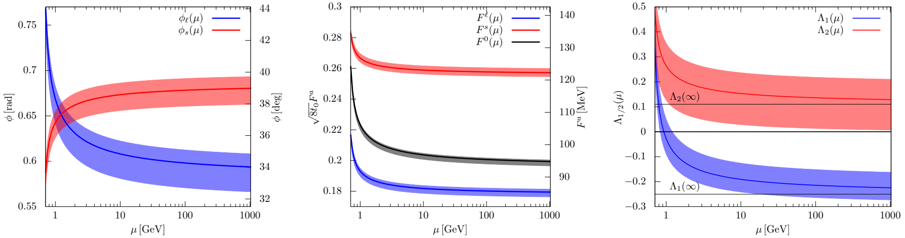
$$F^8 = 115.0\,(^{1.1}_{1.2})_{\rm stat}\,(^{1.6}_{2.4})_{\rm sys}\,(1.5)_{t_0}\,[2.8]_{\rm total}\,\mathrm{MeV} \qquad \theta_8 = -25.8\,(^{1.2}_{2.1})_{\rm stat}\,(^{2.2}_{0.3})_{\rm sys}\,[2.3]^\circ_{\rm total}$$

$$F^0(\mu = 2\,\mathrm{GeV}) = 110.1\,(^{7.0}_{1.9})_{\rm stat}\,(^{2.9}_{2.7})_{\rm sys}\,(1.3)_{t_0}\,[3.0]_{\rm total}\,\mathrm{MeV} \qquad \theta_0 = -8.1\,(^{1.0}_{1.1})_{\rm stat}\,(^{1.5}_{1.5})_{\rm sys}\,[1.8]^\circ_{\rm total}$$

- Masses in agreement with experiment.
- First direct determination of mixing parameters including scale dependence also in quark flavor basis.
- Systematic errors from spread of various fit variations, data cuts ...
- However: $\chi^2/N_{dof} = 179/122 = 1.47$ of best fit still gives $p < 0.001$.

RQCD '21: $U(3)$ χPT LECs and scale dependence

Bali et al., JHEP 08 (2021) 137



Results for LECs:

$M_0(\mu = \infty) = 761\,(^{13}_{21})_{\rm stat}\,(^{18}_{11})_{\rm sys}\,(11)_{t_0}\,[27]_{\rm total}\,\mathrm{MeV}$ $\quad$ $F = 87.71\,(^{1.44}_{1.57})_{\rm stat}\,(^{2.69}_{0.81})_{\rm sys}\,(1.31)_{t_0}\,[2.8]_{\rm total}\,\mathrm{MeV}$

$\Lambda_1(\mu = \infty) = -0.25\,(^{1}_{4})_{\rm stat}\,(^{6}_{2})_{\rm sys}\,[5]_{\rm total}$ $\quad$ $L_5 = +1.66(11)_{\rm stat}(20)_{\rm sys}[23]_{\rm total} \times 10^{-3}$

$\Lambda_2(\mu = \infty) = +0.11\,(^{5}_{5})_{\rm stat}\,(^{7}_{10})_{\rm sys}\,[10]_{\rm total}$ $\quad$ $L_8 = +1.08(09)_{\rm stat}(09)_{\rm sys}[13]_{\rm total} \times 10^{-3}$

- Results agree reasonably well with pheno determinations.
  Leutwyler, NPB Proc. Suppl. 64 (1998) 223 **Benayoun et al.**, EPJ C 17 (2000) 593 **Guo et al.**, JHEP 06 (2015) 175 **Bickert et al.**, PRD 95, 054023 (2017)
- Results for scale-invariant combinations: $M_0/\sqrt{1+\Lambda_1} = 877\,(^{12}_{10})_{\rm stat}\,(^{21}_{8})_{\rm sys}\,(13)_{t_0}\,\mathrm{MeV}$ and $\tilde{\Lambda} = -0.46(19)$.
- Test of Feldmann-Kroll-Stech scheme: **Valid if $\Lambda_1(\mu)$ is small, i.e. for** $0.8\,\mathrm{GeV} \lesssim \mu \lesssim 1.5\,\mathrm{GeV}$.

RQCD '21: Gluonic matrix elements

Bali et al., JHEP 08 (2021) 137

Renormalized gluonic matrix elements $a_P(\mu) = \langle 0 | \omega | P \rangle$ via singlet AWI:

$$\partial_\mu A^0_\mu = \frac{2}{3}(2m_l + m_s)P^0 - \frac{2\sqrt{2}}{3}(m_s - m_l)P^8 + \sqrt{6}\omega$$

from axialvector MEs F^0_P and pseudoscalar MEs $H^{0,8}_P$ $(P = \eta, \eta')$:

$$a_P(\mu) = \sqrt{\frac{2}{3}}\left(M_P^2 F_P^0(\mu) + \frac{2\sqrt{2}}{3}(m_s - m_l)H_P^8 - \frac{2}{3}(2m_l + m_s)H_P^0\right)$$

Physical results:

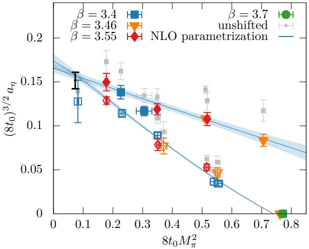
$$a_\eta\ (\mu = 2\,\mathrm{GeV}) = 0.01700\,\left({}^{40}_{69}\right)_{\mathrm{stat}}(48)_{\mathrm{sys}}(66)_{t_0}\ \mathrm{GeV}^3$$
$$a_{\eta'}(\mu = 2\,\mathrm{GeV}) = 0.0381\ \left({}^{18}_{17}\right)_{\mathrm{stat}}(80)_{\mathrm{sys}}(17)_{t_0}\ \mathrm{GeV}^3$$

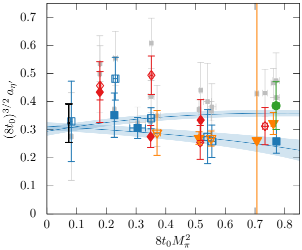
- Mixing angle $\theta_y = \arctan \frac{a_\eta}{a_{\eta'}} = -24.0\left({}^{4.0}_{1.0}\right)_{\mathrm{stat}}(3.2)^\circ_{\mathrm{sys}}$.
- Branching ratio for $J/\psi \to \eta^{(\prime)}\gamma$: NPB 165 (1980) 55-66

$$R(J/\psi) = \frac{\Gamma[J/\psi \to \eta'\gamma]}{\Gamma[J/\psi \to \eta\gamma]} \approx \frac{a^2_{\eta'}}{a^2_\eta} \cdot \left(\frac{k_{\eta'}}{k_\eta}\right)^3 .$$

(assuming anomaly dominates)

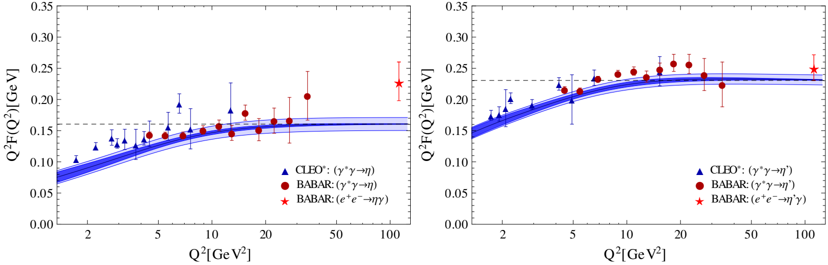
At $\mu = 2\,\mathrm{GeV}$: $R(J/\psi) = 5.03\left({}^{19}_{45}\right)_{\mathrm{stat}}(1.94)_{\mathrm{sys}}$ in agreement with PDG value $R(J/\psi) = 4.74(13)$.





RQCD '21: TFFs at large-Q^2

Bali et al., JHEP 08 (2021) 137



Using results for matrix elements as input for

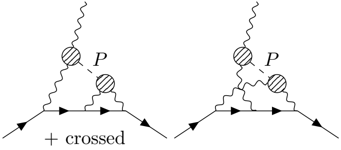
$$\hat{F}_{P\gamma\gamma^*} \equiv \lim_{Q^2\to\infty} F_P = \frac{2}{\sqrt{3}}\left(F_P^8 + 2\sqrt{2}F_P^0(N_f = 4, \mu = \infty)\right), \qquad P = \eta, \eta'$$

$$\boxed{\hat{F}_{\eta\gamma\gamma^*} = 160.5(10.0)\,\mathrm{MeV}, \qquad \hat{F}_{\eta'\gamma\gamma^*} = 230.5(10.1)\,\mathrm{MeV}}$$

- Bands from evaluating QCD predictions (disp. rel. + LCSRs) using lattice results. Agaev et al., PRD 90, 074019 (2014)
- Reasonable agreement with existing experimental data.
- Agreement for $\hat{F}_{\eta\gamma\gamma^*}$ with ETMC '18 result ($\hat{F}_{\eta\gamma\gamma^*} = 155(14)_{\mathrm{stat}}(23)_{\chi PT}\,\mathrm{MeV}$).
- Effect of scale dependence enhanced in $\hat{F}_{\eta'\gamma\gamma^*}$.

$P \to \gamma\gamma$ transition formfactors

$P = \pi_0, \eta, \eta'$ transition formfactors contribute to the LO HLbL scattering in the muon anomalous magnetic moment.



TFF is related to Euclidean P-to-vacuum transition amplitude:

Ji et al., PRL 86, 208 (2001)

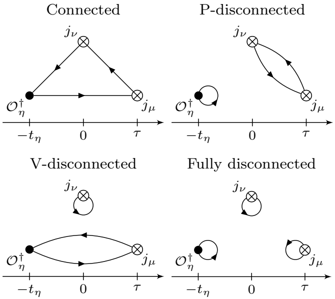
$$\tilde{A}_{\mu\nu}(\tau) \equiv \int d^3\mathbf{x} e^{-i\mathbf{q}_1\cdot\mathbf{x}} \langle 0| T\{j_\mu(\tau,\mathbf{x}) j_\nu(0)\} |P(\mathbf{p})\rangle$$

via

$$\epsilon_{\mu\nu\alpha\beta} q_1^\alpha q_2^\beta \mathcal{F}_{P\to\gamma^*\gamma^*}(q_1^2, q_2^2) = -i^{n_0} \int_{-\infty}^{\infty} d\tau e^{\omega_1 \tau} \tilde{A}^P_{\mu\nu}(\tau),$$

On the lattice: Need to compute three-point functions

$$C_{\mu\nu}(\tau, t_\eta) \equiv \int d^3\mathbf{x}\, d^3\mathbf{y}\, e^{-i\mathbf{q}_1\cdot\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{y}} T\{j_\mu(\tau,\mathbf{x}) j_\nu(0) \mathcal{O}^\dagger_\eta(-t_\eta, \mathbf{y})\}.$$

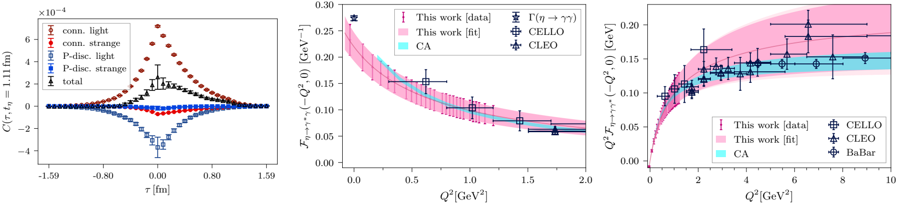


- $j_{\mu,\nu}$ el-mag. currents; $\mathcal{O}_\eta$ interpolating operator(s)
- η (groundstate): $\tilde{A}_{\mu\nu}(\tau) = \lim_{t_\eta \to \infty} \frac{2E_\eta}{Z_\eta} e^{E_\eta t_\eta} C_{\mu\nu}(\tau, t_\eta)$
- η' needs **state projection** (GEVP $\to$ eigenvectors)

Signal2noise problem: only fairly small values of $t_{\eta,\eta'} \lesssim 1\,\mathrm{fm}$ accessible (unlike for π^0).

ETMC '23: $\eta \to \gamma\gamma$ TFF

Alexandrou et al., PRD 108, 054509 (2023)



- Study of $\mathcal{F}_{\eta\to\gamma^*\gamma^*}(Q_1^2, Q_2^2)$ on a physical quark mass ensemble ($a = 0.080\,\mathrm{fm}$) with $N_f = 2+1+1$ tmWilson+Clover quarks.
- Parametrization of TFF by z-expansion **Gerardin et al.**, PRD 100, 034520 (2019)

$$\mathcal{F}^{(z-\exp,N)}_{\eta\to\gamma^*\gamma^*}(-Q_1^2, -Q_2^2) = \left(1 + \frac{Q_1^2 + Q_2^2}{M_V^2}\right)^{-1} \sum_{n,m=0}^{N\le 2} c_{nm} \left(z_1^n - (-1)^{N+n+1}\frac{n}{N+1} z_1^{N+1}\right)\left(z_2^m - (-1)^{N+m+1}\frac{m}{N+1} z_2^{N+1}\right)$$

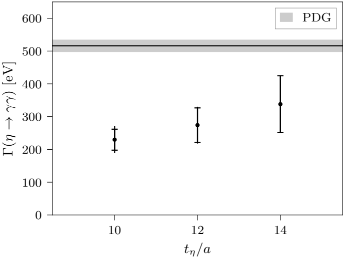
- Results for single-virtual TFF in agreement with experimental data

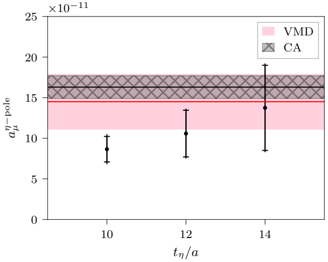
  CELLO: **Behrend et al.**, Z. Phys. C4 9, 401 (1991) CLEO: **Gronberg et al.**, PRD 57, 33 (1998) BABAR: **Aubert et al.**, PRD 80, 052002 (2009)

  and Canterbury approximant estimate **Masjuan et al.**, PRD 95, 054026 (2017)
- However, some tension at the $\sim 2\sigma$ level at small Q^2.

ETMC '23: $\eta\to\gamma\gamma$ TFF - results

Alexandrou et al., PRD 108, 054509 (2023)





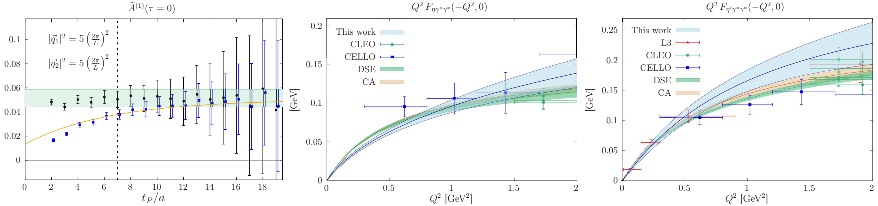
Result for decay width $\Gamma(P\to\gamma\gamma)=\frac{\pi\alpha_{\text{em}}^2 m_P^3}{4}|\mathcal{F}_{P\to\gamma\gamma}(0,0)|^2$: $\Gamma(P\to\gamma\gamma)=338(87)_{\text{stat}}(17)_{\text{sys}}[88]_{\text{total}}\,\text{eV}$

Result for pole contribution to a_μ^{HLbL}: $a_\mu^{\text{HLbL},\eta}=13.8(5.2)_{\text{stat}}(1.5)_{\text{sys}}[5.5]_{\text{total}}\times10^{-11}$

- Systematic errors estimated from fit variations (z-expansion, tail fits)
- Some residual dependence on time-seapration t_η.
- Continuum limit, η': work in progress...

BMW '23: $P \to \gamma\gamma$

Gerardin et al., arXiv:2305.04570



- Study of $\mathcal{F}_{\pi_0,\eta,\eta'\to\gamma^*\gamma^*}(Q_1^2, Q_2^2)$ with $N_f = 2+1+1$ stout-smeared staggered quarks.
- (Near-)physical quark masses, six values of $a \in [0.064 \ldots 0.132]\,\mathrm{fm}$, $L = 3, 4, 6\,\mathrm{fm}$.
- State-projection for transition amplitude $\widetilde{A}^P_{\mu\nu}(\tau) = \sum_{\mathbf{x}} \langle 0|J_\mu(\mathbf{x},\tau)J_\nu(\mathbf{0},0)|n(\mathbf{p})\rangle\, e^{-i\mathbf{q}_1\cdot\mathbf{x}}$

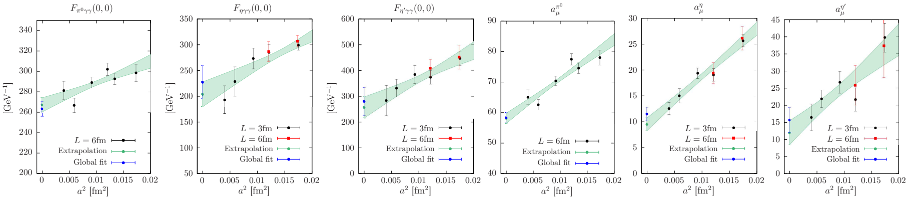
$$\widetilde{A}^{\eta}_{\mu\nu} = \cos^2\phi\,\frac{C^8_{\mu\nu}}{T^8_\eta} + \sin^2\phi\,\frac{C^0_{\mu\nu}}{T^0_\eta}\,, \qquad \widetilde{A}^{\eta'}_{\mu\nu} = \sin^2\phi\,\frac{C^8_{\mu\nu}}{T^8_{\eta'}} + \cos^2\phi\,\frac{C^0_{\mu\nu}}{T^0_{\eta'}}\,, \quad \text{where } T^i_P = \frac{Z^i_P}{2E_P}e^{-E_P(t_f - t_0)}$$

using E_P, Z^i_P ($\tan^2\phi = -(Z^8_{\eta'}Z^0_\eta)/(Z^8_\eta Z^0_{\eta'})$) from a fit to the 2pt correlation function matrix.

However: dedicated spectroscopy analysis yet to be published...

BMW '23: $P \to \gamma\gamma$ - continuum limit

Gerardin et al., arXiv:2305.04570



Physical results for decay widths and pole contributions to a_μ:

$$\Gamma[\pi^0 \to \gamma\gamma] = 7.11(44)_{\rm stat}(21)_{\rm sys}\mathrm{eV}, \qquad a_\mu^{\rm HLbL,\pi} = 57.8(1.8)_{\rm stat}(0.9)_{\rm stat} \times 10^{-11},$$

$$\Gamma[\eta \to \gamma\gamma] = 388(94)_{\rm stat}(35)_{\rm sys}\mathrm{eV}, \qquad a_\mu^{\rm HLbL,\eta} = 11.6(1.6)_{\rm stat}(0.5)_{\rm stat}(1.1)_{\rm FSE} \times 10^{-11},$$

$$\Gamma[\eta' \to \gamma\gamma] = 3.4(1.0)_{\rm stat}(0.4)_{\rm sys}\mathrm{keV}, \qquad a_\mu^{\rm HLbL,\eta'} = 15.7(3.9)_{\rm stat}(1.1)_{\rm stat}(1.3)_{\rm FSE} \times 10^{-11}.$$

Disclaimer: Results not yet published.

$$\boxed{a_\mu^{\rm HLbL,P} = 85.1(4.7)_{\rm stat}(2.3)_{\rm sys} \cdot 10^{-11}}$$

- $\Gamma[\eta \to \gamma\gamma]$ agrees with ETMC, similar tension $\lesssim 2\sigma$ with experiment.
- $a_\mu^{\rm HLbL,\eta}$: some tension with whitepaper estimate $a_\mu^{\rm HLbL,\eta} = 16.3(1.4) \times 10^{-11}$ Masjuan et al., PRD 95, 054026 (2017) mostly due to data at momenta $< 0.5\,\mathrm{GeV}^2$
- Good agreement for $a_\mu^{\rm HLbL,\eta'}$, i.e. $a_\mu^{\rm HLbL,\eta'} = 14.5(1.9) \times 10^{-11}$

Summary and outlook

- LQCD studies of η, η' have made tremendous progress in the last decase, but remain very challenging.
- **Physical extrapolations with controlled systematics have become state-of-the-art.**
- Physical results for masses with $\lesssim 2\%$ error, agreement with experiment
- Matrix elements / mixing parameters and decay constants from first principles with very competitive precision
- Large-N_c χPT can be used to describe / extrapolate lattice data.

  $\rightarrow$ LECs can be determined, including scale dependence

  $\rightarrow$ However, some tension remains...
- Simulations directly at physical quark mass now feasible.

  $\rightarrow$ Direct continuum extrapolations

  $\rightarrow$ Remove need for chiral extrapolation entirely
- First studies beyond masses and mixing: TFFs; more will follow.

Backup slides

Effects of topology in finite volume (I)

In **finite volume** and at **fixed top. charge** Q_t

$$\langle \omega(x)\omega(0)\rangle_{Q_t=\text{fixed}} \to \frac{1}{V}\left(\chi_t - \frac{Q_t^2}{V} + \frac{c_4}{2V\chi_t}\right) + \dots,$$

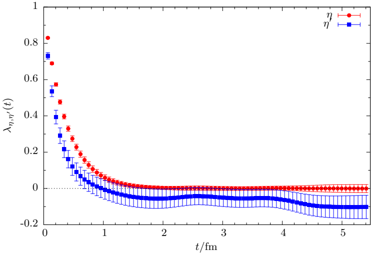
for correlators of winding number densities $\omega(x)$ at large $|x|$.

S. Aoki et al., Phys.Rev. D76, 054508 (2007)

$\Rightarrow$ Expect constant offset in η' (η) correlator **at large** t:

$$< \lambda^{\eta'}(t) >_{Q_t=\text{fixed}} \to \sim \frac{a^5}{T}\left(\chi_t - \frac{Q_t^2}{V} + \frac{c_4}{2V\chi_t}\right).$$

G. S. Bali et al., Phys.Rev. D91 (2015) 1, 014503

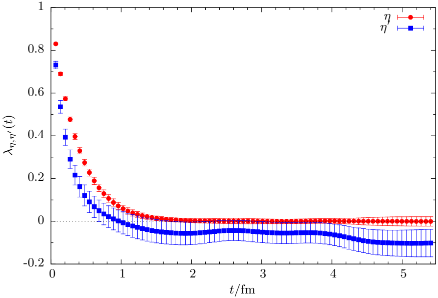


η,η' principal correlators

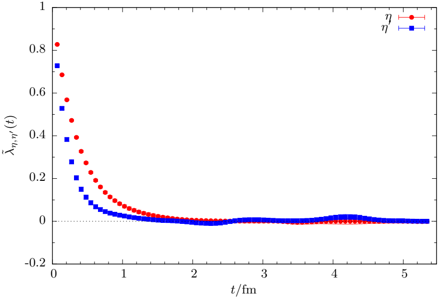
tmWilson+Clover, $M_\pi = 137\,\text{MeV}$, $a = 0.068\,\text{fm}$, $L \approx 5.4\,\text{fm}$

- Always present for finite volume + finite statistics.
- Offset usually compatible with zero within very large statistical point errors.
- **(Correlated) Noise in η'-signal largely due to fluctuation + autocorrelation of this constant.**
- Causes issues for correlated fits / solving GEVPs.

Effects of topology in finite volume (II)



η,η' principal correlators



Same, but from time-derivative

tmWilson+Clover, $M_\pi = 137\,\mathrm{MeV}$, $a = 0.068\,\mathrm{fm}$, $L \approx 5.4\,\mathrm{fm}$

Simple but efficient way to correct for this effect: PRD 97, 054508 (2018)

- Remove constant using discrete time-derivative correlator:

$$\mathcal{C}(t) \to \tilde{\mathcal{C}}(t) = \mathcal{C}(t) - \mathcal{C}(t+\Delta t)$$

- Resulting data are much less correlated, much smaller point errors.
- Further analysis (GEVP, physical extrapolation) can be carried out in the standard way.

However: η' signal still lost around $t \approx 1\,\mathrm{fm}$