Six-Meson Scattering and Three Pions on the Lattice

Chiral Dynamics 2024, Bochum

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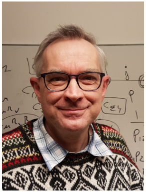
CPT
A*Midex Initiative d'excellence Aix-Marseille
Lund University
Vetenskapsrådet

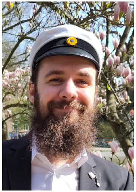
27th August 2024

The people



Hans Bijnens,
Lund U.

Tomáš Husek,
Birmingham U.



Mattias Sjö,
CPT Marseille

Stephen Sharpe,
U. of Washington

Fernando Romero-López,
MIT $\rightarrow$ Bern U.

Jorge Baeza-Ballesteros,
U. de València

The papers

Bijnens & Husek, "*Six-pion amplitude*"

PRD, 2107.06291[hep-ph]

Bijnens, Husek & **MS** "*Six-meson amplitude in QCD-like theories*"

PRD, 2206.14212[hep-ph]

Baeza-Ballesteros, Bijnens, Husek, Romero-López, Sharpe & **MS**
"*The isospin-3 three-particle K-matrix at NLO in ChPT*"

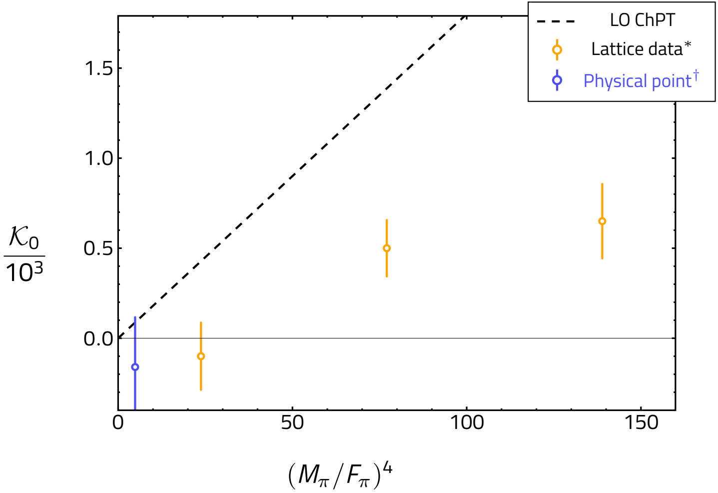
JHEP, 2303.13206[hep-ph]

Baeza-Ballesteros, Bijnens, Husek, Romero-López, Sharpe & **MS**
"*The three-pion K-matrix at NLO in ChPT*"

JHEP, 2401.14293[hep-ph]

- **Introduction**: Fernando's plenary (this morning)
- **Outlook**: ~~Steve's~~ Ferando's parallel talk (18:00)

The tension that was



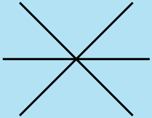
* Blanton, Hanlon, Hörz, Morningstar, Romero-López & Sharpe, "*Three-body interactions from the finite-volume QCD spectrum*" *Phys.Rev.D*, 2021.06144[hep-lat]

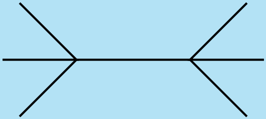
† (see Fernando's talk)

The amplitude

Leading order

Ancient current algebra result





Osborn (1969)

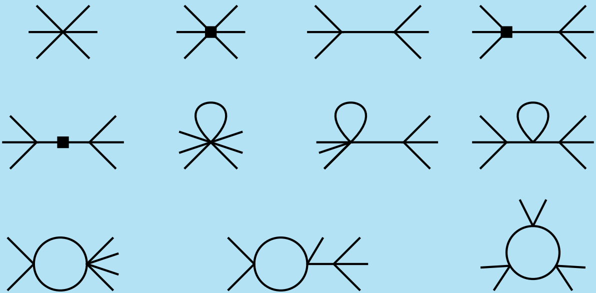
Susskind & Frye (1970)

NLO in CHPT

Vertices

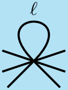
 = LO vertex    = NLO vertex (counterterm)

All the LO and NLO diagrams

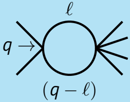


One-Loop Integrals

One- and two-propagator integrals

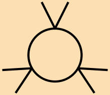


$$\sim \frac{1}{4-d} + (\text{finite})$$



$$\sim \frac{1}{4-d} + \bar{J}(q^2) + (\text{finite})$$

Three-propagator integral



$$\sim \int \frac{d^d\ell}{(2\pi)^d} \frac{\{1, \ell^\mu, \ell^\mu\ell^\nu, \ell^\mu\ell^\nu\ell^\rho\}}{(\ell^2 - M^2)\left[(\ell - q_1)^2 - M^2\right]\left[(\ell + q_2)^2 - M^2\right]}$$

In principle reducible to $\bar{J}$ — **impractical** — redundant basis instead:

$$\{\bar{J}, C, C_{11}, C_{21}, C_3\}(p_1, \ldots, p_6)$$

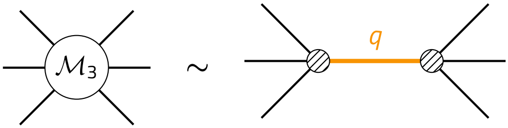
Simplifying the amplitude

$\mathcal{M}_3^{\mathrm{NLO}}$ is a function of...

- 6 particle flavors
- 9 kinematic invariants (8 in $d = 4$)
- 8 free parameters (5 with just pions)
- $\bar{J}(q_i, q_j)$ and 4 $C_X(p_i, p_j, p_k, p_l, p_m, p_n)$'s

$\sim$ **500 pages** in full $\rightarrow$ How to simplify?

Single-particle pole



Factorization

$$\mathcal{M}_3 = \sum_{\substack{\{ijk\} \\ \{lmn\}}} \frac{\mathcal{M}_2(p_i, p_j, p_k, +q) \times \mathcal{M}_2(p_k, p_l, p_n, -q)}{q^2 - M^2 + i\epsilon} + \text{(non-factorizable)}$$

Stripped amplitudes

The 4-point amplitude

$$\begin{aligned}\mathcal{M}^{abcd}(s,t) = & \big[\langle \boldsymbol{abcd}\rangle + \langle dcba\rangle\big]\boldsymbol{B(s,t,u)} + \langle \boldsymbol{ab}\rangle\langle \boldsymbol{cd}\rangle \boldsymbol{C(s,t,u)} \\ & + \big[\langle acdb\rangle + \langle bdca\rangle\big]B(t,u,s) \quad + \langle ac\rangle\langle bd\rangle C(t,u,s) \\ & + \big[\langle adbc\rangle + \langle cbda\rangle\big]B(u,s,t) \quad + \langle ad\rangle\langle bc\rangle C(u,s,t)\end{aligned}$$

The *stripped* 4-point amplitude

$$B = \mathcal{M}_{\{4\}}\,, \qquad C = \mathcal{M}_{\{2,2\}}$$

Stripped amplitudes

Flavour structures

$$\mathcal{F}_{\{6\}}(a_1,\ldots,a_6) = \langle a_1 \cdots a_6 \rangle$$
$$\mathcal{F}_{\{2,4\}}(a_1,\ldots,a_6) = \langle a_1 a_2 \rangle \langle a_3 \cdots a_6 \rangle$$
$$\mathcal{F}_{\{3,3\}}(a_1,\ldots,a_6) = \langle a_1 a_2 a_3 \rangle \langle a_4 a_5 a_6 \rangle$$
$$\mathcal{F}_{\{2,2,2\}}(a_1,\ldots,a_6) = \langle a_1 a_2 \rangle \langle a_3 a_4 \rangle \langle a_5 a_6 \rangle$$

$$\mathcal{M}(p_1, a_1; p_2, a_2; \ldots) = \sum_R \sum_\sigma \mathcal{M}_R(\sigma[p_1,\ldots])\mathcal{F}_R(\sigma[a_1,\ldots])$$

Stripping

$\sigma \notin$ symmetries of $\mathcal{F}_R$
$\rightarrow$ well-known, unique

Deorbiting

$\sigma \in$ symmetries of $\mathcal{F}_R$
$\rightarrow$ novel, non-unique!

$\mathcal{M}_3^{\mathrm{NLO}}$ still won't fit on a slide, but not far from it!

The K-matrix

How to get it

$\mathcal{K}_{\text{df},3}$ ← Unspeakably horrifying integral equation → $\mathcal{M}_3$

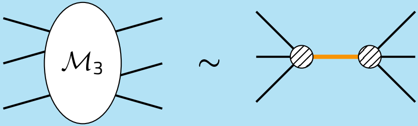
$\mathcal{K}^{\text{NLO}}_{\text{df},3}$ ← $\mathcal{K} = \mathrm{Re}(\mathcal{M} - \mathcal{D})$ → $\mathcal{M}^{\text{NLO}}_3$

Guiding properties of $\mathcal{D}$

- Exactly cancels divergences $\rightarrow$ **df**
- All internal lines on-shell

Subtracting poles

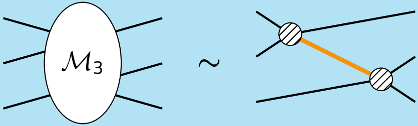
s-channel one-particle exchange



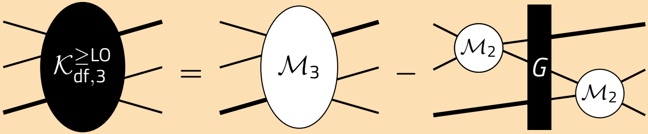
- Only present at **isospin 1**
- **No subtraction** needed since pole is sub-threshold

Subtracting poles

t-channel one-particle exchange



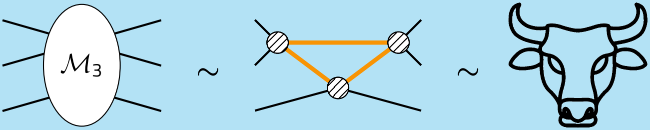
OPE subtraction



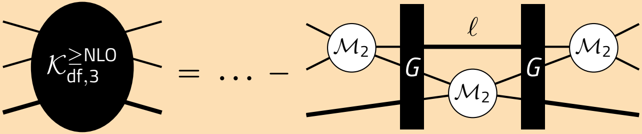
Note: G necessarily contains a cutoff function

Subtracting cuts

Bull's head cut



Bull's head subtraction



The bull's head

The bull's head integral is **awful**:

- Triangle loop $\Rightarrow$ complicated, pole-ridden integrand
- On-shell $\Rightarrow$ no loop momentum shift
- Non-analytic $\Rightarrow$ no Wick rotation, dispersion, etc.

Different approaches

- Some clever but not very useful ways
  Simple part with poles + complicated part (numerics-friendly)
- Brute-force numerics
  Because Tomáš is a Mathematica wizard
- Semi-analytic
  Threshold-expand, then apply deep magic

Threshold expansion

Expansion parameters

$$\Delta \propto P^2 - (3M_\pi)^2$$ (**system** above-threshold-ness)

$$\Delta_i^{(\prime)} \propto (P - p_i^{(\prime)})^2 - (2M_\pi)^2$$ (**pair** above-threshold-ness)

$$\tilde{t}_{ij} \propto (p_i - p_j')^2$$ (**spectator** above-threshold-ness)

Compound parameters

$$\Delta_\mathrm{A} = \sum (\Delta_i^2 + \Delta_i'^2) - \Delta^2 \qquad \Delta_\mathrm{B} = \sum \tilde{t}_{ij}^{\,2} - \Delta^2$$

Maximum isospin threshold expansion

$$\mathcal{K}_{\mathrm{df},3}^{[I=3]} = \mathcal{K}_0 + \mathcal{K}_1 \Delta + \mathcal{K}_2 \Delta^2 + \mathcal{K}_\mathrm{A} \Delta_\mathrm{A} + \mathcal{K}_\mathrm{B} \Delta_\mathrm{B} + \mathcal{O}(\Delta^3)$$

Non-maximal isospin

$I = 3$ — Singlet

$I = 2$ — Doublet

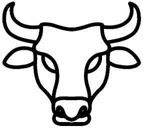
$I = 1$ — Singlet, Doublet

$I = 0$ — Antisymmetric singlet

Minimum isospin threshold expansion

$$\mathcal{K}_{\text{df},3}^{[I=0]} = \mathcal{K}_0^{\text{AS}} \sum \epsilon_{ijk}\epsilon_{lmn}\tilde{t}_{il}\tilde{t}_{jm} + \mathcal{O}(\Delta^3)$$

Semi-analytic evaluation



Expand → $\sum \int \mathrm{d}x \, [\text{End-point divergent}](x)$

↓ **Hadamard partie finie***

$\sum \int \mathrm{d}x \, [\text{Finite}](x)$

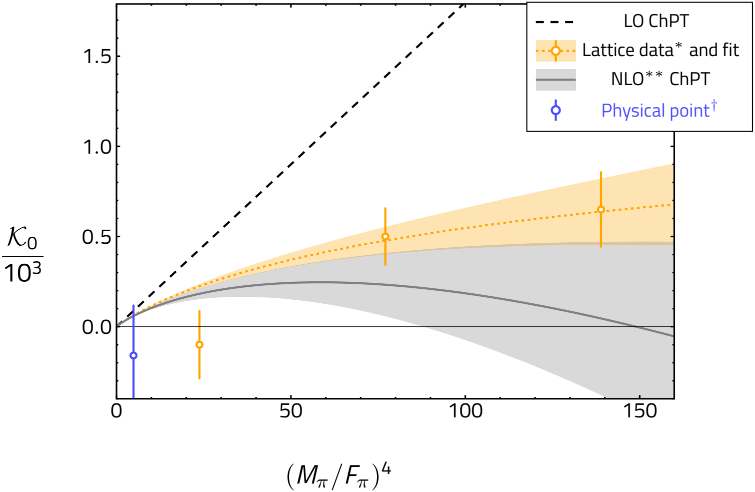
↓

$$\underbrace{\sum \int \mathrm{d}x \, [\text{Simple}](x)}_{\text{analytic}} + \underbrace{\int \mathrm{d}x \, [\text{Small}](x)}_{\substack{\text{numeric} \\ \text{cutoff-dependent}}}$$

* Costin & Friedman, "*Foundational aspects of singular integrals*"
J.Functional Analysis, 1401.7045[math.FA]

Results

Maximum isospin, again



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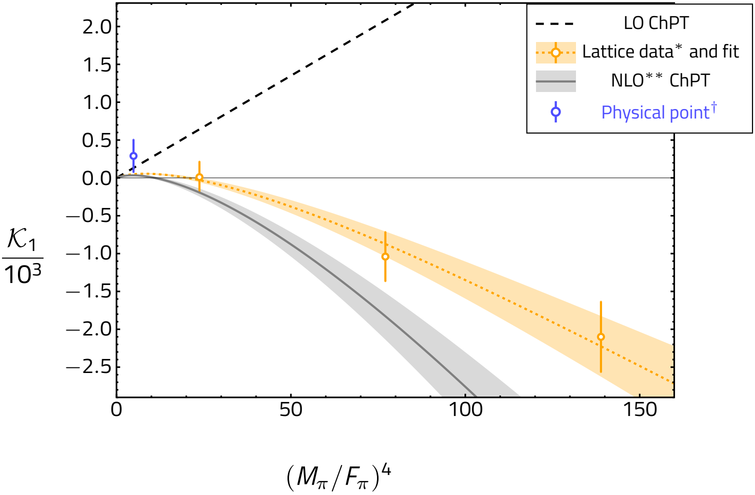
** using LECs from FLAG and Colangelo, Gasser & Leutwyler, "*$\pi\pi$ scattering*" *Nucl.Phys.B*, hep-ph/0103088

† (see Fernando's talk)

Subleading order



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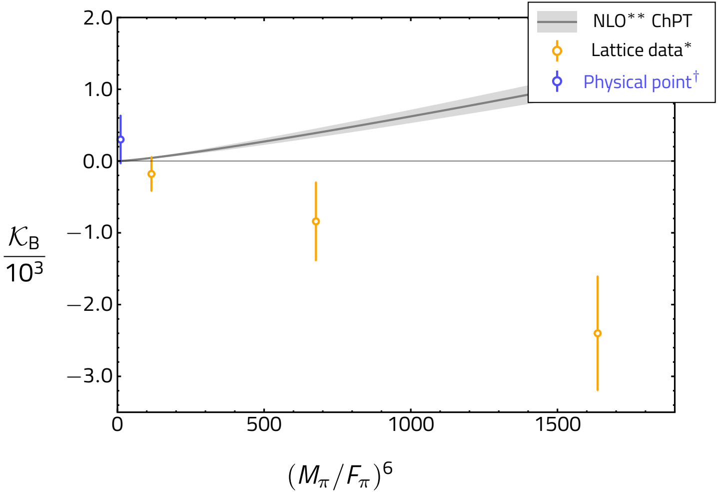
Sub-subleading order



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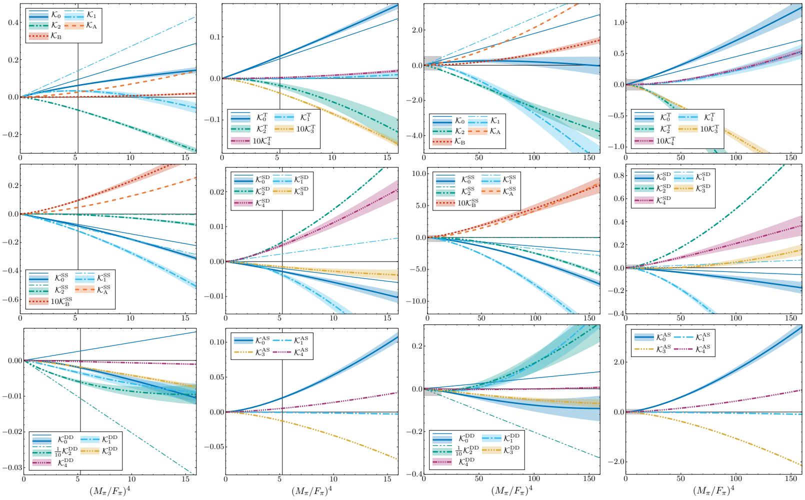
Awaiting more lattice results...



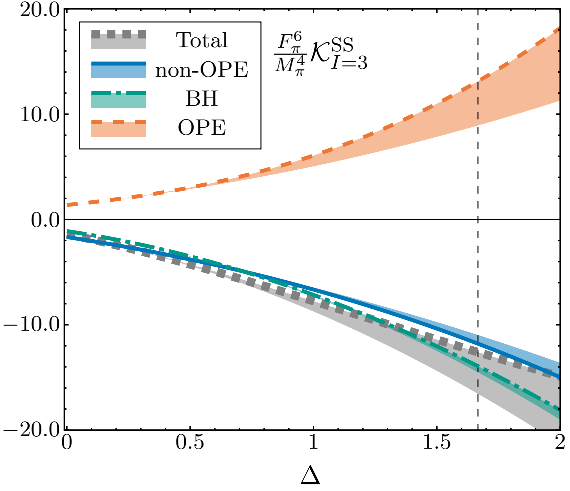
Summary

Summary

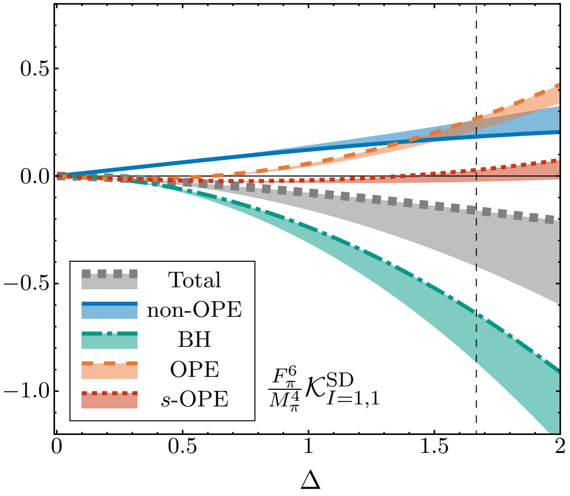
- All three-pion channels covered
- Main tension resolved
  (where lattice data are available)
- Next steps: More flavors, eventually nucleons
  Maybe 2 loops? (if anyone ever needs it)
  Maybe even 4 particles? (if anyone ever needs it)
  See also ~~Steve's~~ Fernando's parallel talk (18:00)
- Pion-kaon case is underway on the amplitude side

Convergence

The threshold expansion works



...better than it has to



Does ChPT converge?

- Large LO-NLO difference is troubling...
- ...but LO is very constrained
  $\Rightarrow$ **qualitative** difference expected
- Adding NNLO: **extremely difficult**:
  - Two-loop 6-point amplitude
  - Integral relation between $\mathcal{M}_3$ and $\mathcal{K}_{\text{df},3}$