Weak decays and finite volume QED

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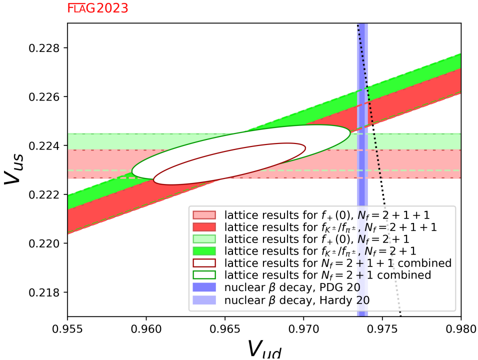


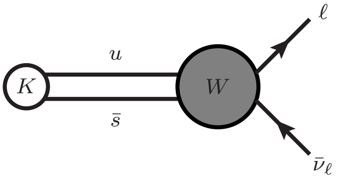
- Determine $|V_{ud}|$ and $|V_{us}|$
- Tensions with unitarity?
- 1.7–5 σ
- Depends on the input!
- Need to control theory/exp. uncertainties

- Precision goal: (Sub-)% level
- Isospin-breaking effects crucial:

$$\text{QED: } \alpha \neq 0$$
$$\text{QCD: } m_u \neq m_d$$

- Consider diagonal $|V_{us}|/|V_{ud}|$: Leptonic decays

- Access $|V_{us}|/|V_{ud}|$ from leptonic kaon/pion decays



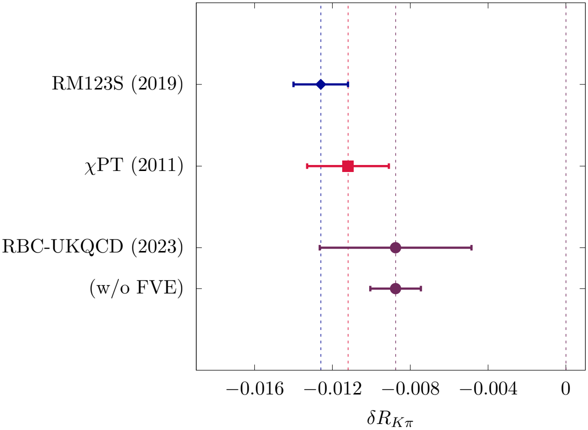
$$\underbrace{\Gamma\left(K^- \to \mu^- \nu_\mu\right)}_{\text{Exp.}} \propto |V_{us}|^2 \underbrace{\frac{\left(m_K^2 - m_\mu^2\right)^2}{m_K^3}}_{\text{Exp.}} \underbrace{f_K^2 \left(1 + \delta R_K\right)}_{\text{Theory}}$$

- Combine experiment and theory (lattice)

$$\frac{|V_{us}|^2}{|V_{ud}|^2} = \frac{\text{Kaon exp.}}{\text{Pion exp.}} \times \frac{f_K^2}{f_\pi^2}\left(1 + \delta R_K - \delta R_\pi\right)$$

- Isospin-breaking corrections in $\delta R_K - \delta R_\pi$: % level precision

[RM123S 2019; Di Carlo, Hansen, NHT, Portelli 2022; RBC/UKQCD 2023]



RBC-UKQCD 23:

$\delta R_{K\pi} = -0.0086(39)$

total	(39)
total (w/o FVE)	(13)
statistical	(3)
FVE	(37)
fit	(11)
QED quenching	(5)
discretisation	(5)

χPT : $\delta R_{K\pi} = -0.0112(21)$

RM123S 19: $\delta R_{K\pi} = -0.0126(14)$

Issue: QED finite-volume effects not sufficiently understood

Simulate at different volumes: Analytical knowledge important

- Correct δR_P data with $Y(L)$

$$Y(L) = \underbrace{Y_0 + Y_{\log}\log(m_P L) + \frac{Y_1}{m_P L}}_{\text{[RM123/S 16]}} \underbrace{+ \frac{Y_2}{(m_P L)^2} + \frac{Y_3}{(m_P L)^3}}_{\text{[Di Carlo, Hansen, NHT, Portelli 22/23]}} + \ldots$$

- Universal: $\log + 1/L$
- Structure dependent: $1/L^2 + 1/L^3$

$$Y_3 = \frac{32\pi^2\, m_P}{f_P(1-r_\ell^4)}\left\{ c_0(\mathbf{v}_\ell)\,\left[F_V^P - F_A^P + 2m_P^2 r_\ell^2\, F_A^{P\,\prime}\right] + c_0\, \mathcal{C}_\ell \right\}$$

- F_V^P, F_A^P, $F_A^{P\,\prime}$ Lattice [RM-123/S 20/22/23, RBC/UKQCD 23], ChPT [Bijnens et al. 92]
- Finite-volume coefficients $c_j(\mathbf{v}_\ell)$ at every order
- Cannot determine structure dependent $\mathcal{C}_\ell$: Remove?
- $Y_3 \approx Y_3^{\mathrm{pt}}$ numerically large $\longrightarrow$ drove our FV uncertainty

- Clearly we need a better understanding of finite-volume effects
- Gauss' law: Difficult to define charged states in finite volume with periodic boundary conditions

$$Q = \int_V d^3\mathbf{x}\, \nabla \cdot \mathbf{E}(t, \mathbf{x}) = \int_{\partial V} d\mathbf{S} \cdot \mathbf{E} = 0$$

- Problem: Photon zero-momentum modes and absence of mass gap
- Define QED in a finite volume: $\mathrm{QED_L}$ above (others available)
- $\mathrm{QED_L^{IR}}$: Exclude/redistribute photon zero-mode
  [Davoudi, Harrison, Jüttner, Portelli, Savage 2019]
  - $\mathrm{QED_L}$: Exclude photon zero-mode [Hayakawa, Uno 2008]
  - $\mathrm{QED_r}$: Redistribute photon zero-mode [Di Carlo, Hansen, NHT, Portelli in prep.]

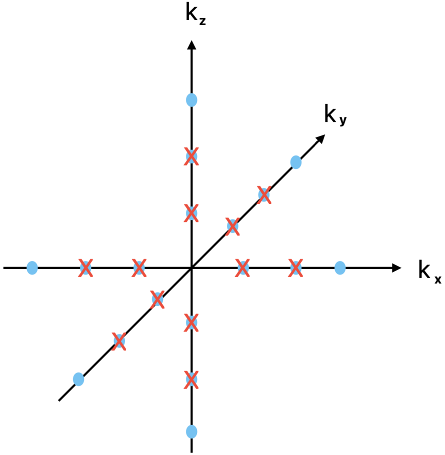
- Photon momentum $k = (k_0, \mathbf{k})$
- 3-momentum discretised by finite volume extent $\mathbf{k} = \frac{2\pi\mathbf{n}}{L}$
- Allows definition of finite-volume QED: $\mathbf{k} \neq \mathbf{0}$
- General form of propagator: Prescription dependent $\mathrm{QED}_{\mathrm{L}}^{\mathrm{IR}}$

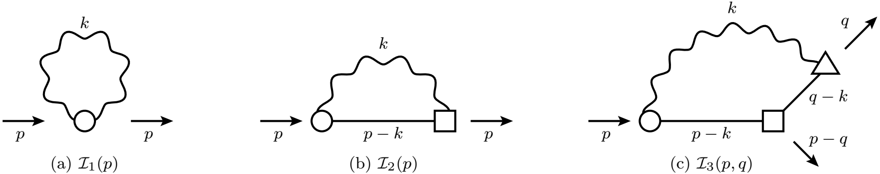
$$D_{\mu\nu}(k) = \delta_{\mu\nu}\,\frac{1 + w_{|\mathbf{n}|^2}}{k^2}$$

- Weights $w_{|\mathbf{n}|^2}$ in action [Davoudi, Harrison, Jüttner, Portelli, Savage 2019]
- $\mathrm{QED}_{\mathrm{L}}^{\mathrm{IR}}$ and $\mathrm{QED}_{\mathrm{r}}$ non-zero for finite number of $\mathbf{n} = \mathbf{k}\,L/(2\pi)$
- $\mathrm{QED}_{\mathrm{L}}$ $w_{|\mathbf{n}|^2} = 0$

What does this mean?

- $\mathbf{k} = \frac{2\pi\mathbf{n}}{L}$ where $\mathbf{n} \neq \mathbf{0}$
- Add weights $w^2_{|\mathbf{n}|}$ on finite number of modes
- Example: Inner two shells
- $w^2_{|\mathbf{n}|=1}$ and $w^2_{|\mathbf{n}|=2}$
- Can we use the freedom to our advantage?





(a) $\mathcal{I}_1(p)$ (b) $\mathcal{I}_2(p)$ (c) $\mathcal{I}_3(p,q)$

- Only prescription dependence in finite-volume coefficients

$$c_j^w(\mathbf{v}) = \left(\sum_{\mathbf{n}\neq\mathbf{0}} - \int d^3\mathbf{n} \right) \frac{1 + w_{|\mathbf{n}|^2}}{|\mathbf{n}|^j \left(1 - \mathbf{v}\cdot\hat{\mathbf{n}}\right)}$$

- Note in particular that

$$c_j^w(\mathbf{v}) = \underbrace{c_j(\mathbf{v})}_{\mathrm{QED_L}} + \sum_{\mathbf{n}\neq\mathbf{0}} \frac{w_{|\mathbf{n}|^2}}{|\mathbf{n}|^j \left(1 - \mathbf{v}\cdot\hat{\mathbf{n}}\right)}$$

- Can we remove the problematic part $c_0^w\, \mathcal{C}_\ell/(m_P L)^3$?

Minimal choice: $\mathrm{QED_r}$

- Want $c_0^w = 0$

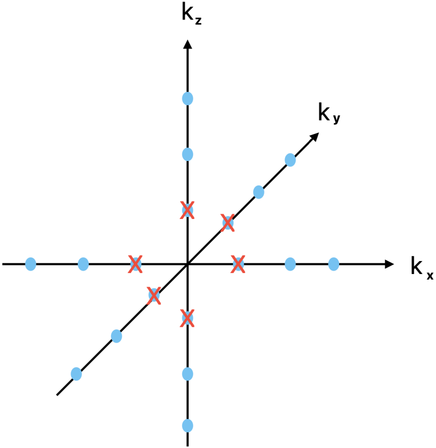
$$c_0^w = \underbrace{c_0}_{=-1} + \sum_{|\mathbf{n}|} w_{|\mathbf{n}|^2}$$

- Solution: $w_{|\mathbf{n}|^2} = \delta_{|\mathbf{n}|,1}/6$

$$\overline{c}_0 = 0$$

- Why it works: $|\mathbf{k}| \propto 1/L$
- Generally:

$$\overline{c}_j(\mathbf{v}) = \underbrace{c_j(\mathbf{v})}_{\mathrm{QED_L}} + \frac{1}{6} \sum_{|\mathbf{n}|=1} \frac{1}{(1-\mathbf{v}\cdot\hat{\mathbf{n}})}$$



- Our formula is now $(\mathrm{QED_r})$:

$$\overline{Y}(L) = \overline{Y}_0 + \overline{Y}_{\log}\,\log(m_P L) + \frac{\overline{Y}_1}{m_P L} + \frac{\overline{Y}_2}{(m_P L)^2} + \frac{\overline{Y}_3}{(m_P L)^3} + \overline{\cdots}$$

$$\overline{Y}_3 = \frac{32\pi^2\, m_P}{f_P(1-r_\ell^4)}\,\overline{c}_0(\mathbf{v}_\ell)\,\left[F_V^P - F_A^P + 2m_P^2 r_\ell^2\, F_A^{P\,\prime}\right]$$

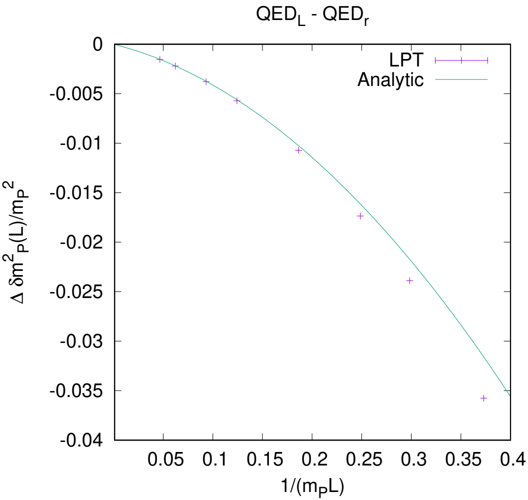
- Our problem is solved!
- Current study: Asymptotic behaviour of $\frac{\bar{c}_j}{L^{3+|j|}}$ for large $|j|$
- Can we do more?
- What about $\overline{c}_0(\mathbf{v}_\ell)$? Magic velocity $\overline{c}_0(\mathbf{v}_\ell^{\mathrm{magic}}) = 0$

Conclusions and outlook

- Searching for/constraining new physics with precision calculations
- Systematic finite-volume effects crucial
- Leptonic decays: Solved bottleneck, $\mathrm{QED_r}$
- New RBC-UKQCD calculation with $\mathrm{QED_r}$
- Address volume and sub-leading uncertainties
- $\mathrm{QED_r}$: General, also other processes
- Semi-leptonic decays, $\pi\pi$ scattering, ...

Backup slides

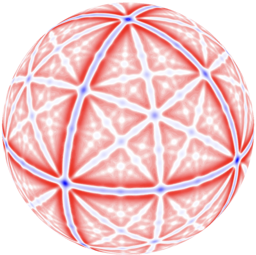
$$
\begin{aligned}
Y^{(3)}(L) = {}& \frac{3}{4} + 4\,\log\left(\frac{m_\ell}{m_W}\right) + \frac{\overline{c}_3 - 2\,\overline{c}_3(\mathbf{v}_\ell)}{2\pi} - 2\,A_1(\mathbf{v}_\ell) + 2\,\log\left(\frac{m_W L}{4\pi}\right) \\
& - 2\,A_1(\mathbf{v}_\ell)\left[\log\left(\frac{m_P L}{4\pi}\right) + \log\left(\frac{m_\ell L}{4\pi}\right)\right] - \frac{1}{m_P L}\left[\frac{(1+r_\ell^2)^2\,\overline{c}_2 - 4\,r_\ell^2\,\overline{c}_2(\mathbf{v}_\ell)}{1-r_\ell^4}\right] \\
& + \frac{1}{(m_P L)^2}\left[-\frac{F_A^P}{f_P}\,\frac{4\pi\,m_P\,\left[(1+r_\ell^2)^2\,\overline{c}_1 - 4\,r_\ell^2\,\overline{c}_1(\mathbf{v}_\ell)\right]}{1-r_\ell^4} + \frac{8\pi\,\left[(1+r_\ell^2)\,\overline{c}_1 - 2\,\overline{c}_1(\mathbf{v}_\ell)\right]}{(1-r_\ell^4)}\right] \\
& + \frac{32\pi^2\,m_P}{f_P(1-r_\ell^4)\,(m_P L)^3}\,\overline{c}_0(\mathbf{v}_\ell)\,\left[F_V^P - F_A^P + 2m_P^2 r_\ell^2\,A^{(0,1)}\left(0,-m_P^2\right)\right]
\end{aligned}
$$



- Lattice perturbation theory
- Scalar QED (pointlike)
- vegas: $\Delta = \sum_{\mathbf{k}\neq\mathbf{0}} - \int_{\mathbf{k}}$
- Captures exponentials

$$\frac{\Delta^{\mathrm{QED_L}} m_P^2(L) - \Delta^{\mathrm{QED_r}} m_P^2(L)}{m_{P,0}^2} \overset{\text{point}}{=} \frac{-1}{4\pi^2(m_P L)} + \frac{-1}{2\pi(m_P L)^2} + \mathcal{O}[e^{-m_P L}]$$

- Let us look at angular dependence in $\bar{c}_0(\mathbf{v})$ for $|\mathbf{v}| = 0.999$ ($\approx D_s \to \mu\nu$ decay)



$\max \bar{c}_0(\mathbf{v}) \approx 9000$

$\min \bar{c}_0(\mathbf{v}) \approx -800$

Figure from Di Carlo, Lattice 2023

- Divergences but there are magic angles: $\bar{c}_0(\mathbf{v}_{\text{magic}}) = 0$
- Angular average: $\int d\Omega\, \bar{c}_0(\mathbf{v}) \propto \bar{c}_0 = 0$ [Davoudi, Harrison, Jüttner, Portelli, Savage 2019]
- Requires many velocities
- Magic velocity seems a reasonable way forward

Comparison of finite-volume coefficients

- $\mathbf{v} = |\mathbf{v}|\,(1,1,1)/\sqrt{3}$: $|\mathbf{v}| = 0.912401$ (corresponds to $K \to \mu\nu_\mu$)

$\mathrm{QED_L}$: $c_j(\mathbf{v})$ $\mathrm{QED_r}$: $\bar{c}_j(\mathbf{v})$ $\mathrm{QED_C}$: $c_j^*(\mathbf{v})$

j	$c_j(\mathbf{v})$	$\bar{c}_j(\mathbf{v})$	$c_j^*(\mathbf{v})$	c_j	$\bar{c}_j$	c_j^*
2	-16.3454	-14.9613	-3.20674	-8.91363	-7.91363	-5.49014
1	-5.73018	-4.3461	3.51224	-2.8373	-1.8373	-0.80194
0	-2.12369	-0.7396	3.69273	-1	0	0

- Effects of non-locality at $1/L^3$ can be removed in $\mathrm{QED_r}$: $\bar{c}_0 = 0$
- $c_0^*(\mathbf{v})$ can give a $1/L^3$ also in $\mathrm{QED_C}$
- $\bar{c}_0(\mathbf{v})$ still gives a $1/L^3$ contribution for leptonic decays