Application of Bayesian statistics to the sector of decay constants in three-flavour χPT

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χPT Lagrangian by Gasser, Leutwyler (1985) [2]:

$$\mathcal{L}_{eff} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \mathcal{L}^{(6)} + \ldots$$

$$\mathcal{L}^{(2)}_{eff} = \frac{F_0^2}{4} Tr[D_\mu U D^\mu U^+ + (U^+\chi + \chi^+ U)]$$

$$\begin{aligned}\mathcal{L}^{(4)}_{eff}(L_1 \ldots L_{10}) = {} & L_1\, Tr[D_\mu U^+ D^\mu U]^2 + L_2\, Tr[D_\mu U^+ D_\nu U]\, Tr[D^\mu U^+ D^\nu U] + \\ & + L_3\, Tr[D_\mu U^+ D^\mu U D_\nu U^+ D^\nu U] + \\ & + L_4\, Tr[D_\mu U^+ D^\mu U]\, Tr[\chi^+ U + \chi U^+] + \\ & + L_5\, Tr[D_\mu U^+ D^\mu U(\chi^+ U + U^+\chi)] + L_6\, Tr[\chi^+ U + \chi U^+]^2 + \\ & + L_7\, Tr[\chi^+ U - \chi U^+]^2 + L_8\, Tr[\chi^+ U \chi^+ U + \chi U^+ \chi U^+] + \\ & + iL_9\, Tr[F_R^{\mu\nu} D_\mu U D_\nu U^+ + F_L^{\mu\nu} D_\mu U^+ D_\nu U] + L_{10}\, Tr[U^+ F_R^{\mu\nu} U F^L_{\mu\nu}]\end{aligned}$$

$$U(x) = \exp\left(\frac{i}{F_0}\phi^a(x)\lambda^a\right) \qquad \chi = 2B_0\,\mathrm{diag}(m_u, m_d, m_s)$$

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Decay constants

Decay constants of light pseudoscalar meson nonet in terms of QCD axial-vector currents:

$$i p_\mu F_P^a = \langle 0 \,|\, A_\mu^a(0) \,|\, P, p \rangle,$$

where $A_\mu^a = \bar{q}\gamma_\mu\gamma_5\lambda^a q$.

$SU(3)$ F_η defined

$$F_\eta = F_\eta^8 = F_8 \cos\vartheta_8$$

Lots of results in literature, we use:

lattice	$F_\eta^8 = (1.123 \pm 0.035) F_\pi$	(RQCD21) [3]
phenomenology	$F_\eta^8 = (1.18 \pm 0.02) F_\pi$	(EGMS15) [4]
	$F_\eta^8 = (1.38 \pm 0.05) F_\pi$	(EF05) [5]

These decay constants as expressed in [6]:

$$
\begin{aligned}
F_\pi^2 &= F_0^2(1 - 4\mu_\pi - 2\mu_K) + 8m_\pi^2\left(L_4^r(r+2) + L_5^r\right) + F_\pi^2\delta_{F_\pi} \\
F_K^2 &= F_0^2\left(1 - \frac{3}{2}\mu_\pi - 3\mu_K - \frac{3}{2}\mu_\eta\right) + 8m_\pi^2\left(L_4^r(r+2) + \frac{1}{2}L_5^r(r+1)\right) + F_K^2\delta_{F_K} \\
F_\eta^2 &= F_0^2(1 - 6\mu_K) + 8m_\pi^2\left(L_4^r(r+2) + \frac{1}{3}L_5^r(2r+1)\right) + F_\eta^2\delta_{F_\eta},
\end{aligned}
$$

where

$$
\begin{aligned}
\mu_P &= \frac{m_P^2}{32\pi^2 F_0^2}\ln\left(\frac{m_P^2}{\mu^2}\right) \\
m_\pi^2 &= 2B_0\hat{m}, \\
m_K^2 &= B_0\hat{m}(r+1), \\
m_\eta^2 &= \frac{2}{3}B_0\hat{m}(2r+1)
\end{aligned}
$$

It is convenient to introduce reparametrization of the chiral order parameters F_0 and B_0:

$$X = \frac{2\,\hat{m}F_0^2 B_0}{F_\pi^2 M_\pi^2} \equiv \frac{2\,\hat{m}\Sigma_0}{F_\pi^2 M_\pi^2}$$

$$Z = \frac{F_0^2}{F_\pi^2}$$

$$Y = \frac{X}{Z} = \frac{2B_0\hat{m}}{M_\pi^2} = \frac{m_\pi^2}{M_\pi^2}$$

Standard approach assumes these values to be close to one:
$\longrightarrow$ LO terms dominating the expansion

Paramagnetic inequality [7] - $X \equiv X(3) < X(2)$, $Z \equiv Z(3) < Z(2)$

If $X = 0$, the chiral condensate would vanish
If $Z = 0$, restoration of chiral symmetry

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Way to accomodate irregular convergence of chiral expansion

Procedure can be described as follows:

- Standard χPT Lagrangian, based on power counting $m_q \sim O(p^2)$
- "safe observables" = related linearly to Green functions of QCD currents
- For an observable A the 'resummed' chiral expansion has the form

$$A = A^{(LO)} + A^{(NLO)} + A\delta A, \qquad \delta A \ll 1$$

- Higher order remainders not neglected, but treated as sources of error
- All higher order LECs are effectively contained in the remainders
- All manipulations in nonperturbative algebraic way

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Bayesian approach

Bayesian approach with parameters X_i and observables O_k can be written as

$$
\begin{aligned}
P(X_i|\mathrm{data}) &= \frac{P(\mathrm{data}|X_i)P(X_i)}{\int \mathrm{d}X_i \, P(\mathrm{data}|X_i)P(X_i)} \\
P(\mathrm{data}|X_i) &= \prod_k \frac{1}{\sigma_k \sqrt{2\pi}} \exp\left[-\frac{(O_k^{exp} - O_k^{theory}(X_i))^2}{2\sigma_k^2}\right]
\end{aligned}
$$

In our case

- Three observables - decay constants F_π, F_K, F_η
- leading order parameters: X, Z
- next-to-leading order parameters: L_4^r, L_5^r
- higher order remainders: δ_{F_π}, δ_{F_K}, δ_{F_η}

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Assumptions

For X, Y, Z we use the same constraints as Kolesár, Novotný (2018) [8]

$$0 < Y < Y_{\max} \simeq 2.5 \qquad 10^3 L_5^r \in (0, 2)$$

$$Y_{\max} = \frac{8F_K^2 M_K^2 (\delta_{M_K} - 1) - 2F_\pi^2 M_\pi^2 (r+1)^2 (\delta_{M_\pi} - 1)}{M_\pi^2 (r+1) (2F_K^2 (\delta_{F_K} - 1) - F_\pi^2 (r+1)(\delta_{F_\pi} - 1))} \qquad 10^3 L_4^r \in (-0.5, 2)$$

$$0 < Z < Z(2) = 0.86 \pm 0.01 \qquad \delta_{F_P} = \delta_{M_P} = 0.0 \pm 0.1$$

$$0 < X < X(2) = 0.89 \pm 0.01$$

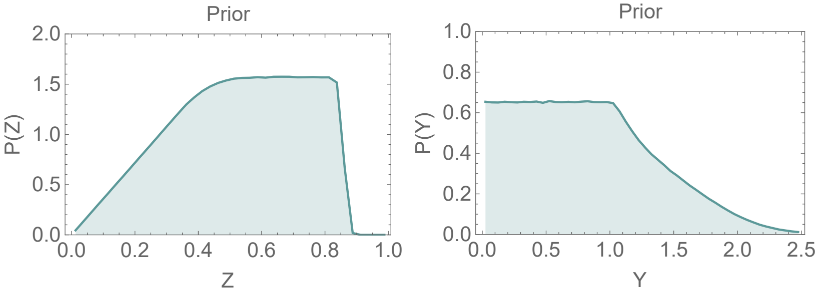

Figure: Prior distributions

To obtain constraints on the NLO LECs L_4^r and L_5^r
$\longrightarrow$ input Y from $\eta \to 3\pi$ decays Kolesár, Novotný (2018) [8]

$$Y = 1.44 \pm 0.32 \qquad (\eta \to 3\pi).$$

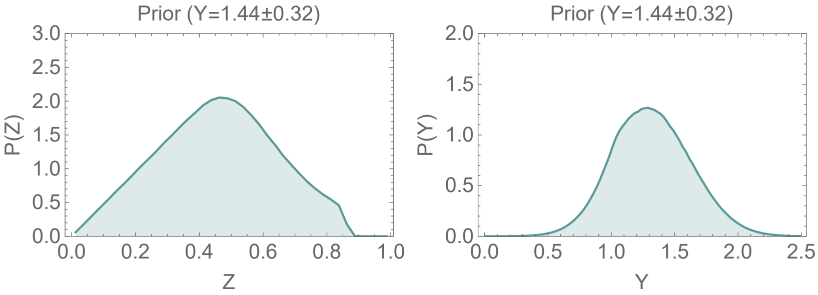

Figure: Prior distributions - Kolesár, Novotný (2018) [8]

As an alternative, we will use χQCD Collaboration (2021)[9] results:

$$Y = 0.95 \pm 0.10,$$
$$Z = 0.54 \pm 0.05 \qquad (\chi\mathrm{QCD21}).$$

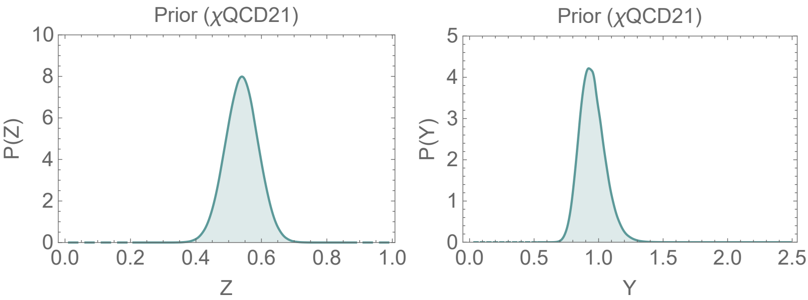

Figure: Prior distributions - [9] (χQCD21).

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Algebraically eliminate F_0, L_4^r and L_5^r $\longrightarrow$ obtain one equation:

$$F_\eta^2 = \frac{1}{3}\left[4F_K^2 - F_\pi^2 + \frac{M_\pi^2 Y}{16\pi^2}\left(\ln\frac{m_\pi^2}{m_K^2} + (2r+1)\ln\frac{m_\eta^2}{m_K^2}\right) + 3F_\eta^2\delta_{F_\eta} - 4F_K^2\delta_{F_K} + F_\pi^2\delta_{F_\pi}\right].$$

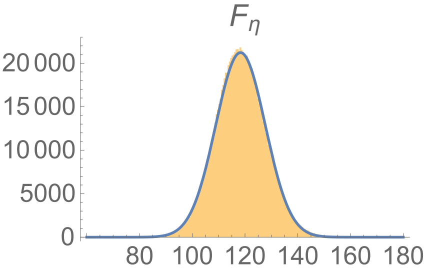

Figure: Theoretical prediction for F_η (10^6 points)

$$F_\eta = 117.5 \pm 9.4 \text{ MeV} = (1.28 \pm 0.10)F_\pi$$
$$F_\eta = 117.7 \pm 9.3 \text{ MeV} \quad (\chi\text{QCD21})$$

Use RQCD21 [3], EGMS15 [4] and EF05 [5] as alternative inputs to extract information about the remainders:

$$\begin{aligned} \delta_{F_K} &= 0.10 \pm 0.07, & \delta_{F_\eta} &= -0.08 \pm 0.08, & \rho &= 0.71 & &(\mathrm{RQCD21}),\\ \delta_{F_K} &= 0.07 \pm 0.06, & \delta_{F_\eta} &= -0.06 \pm 0.08, & \rho &= 0.85 & &(\mathrm{EGMS15}),\\ \delta_{F_K} &= -0.06 \pm 0.08, & \delta_{F_\eta} &= 0.05 \pm 0.08, & \rho &= 0.64 & &(\mathrm{EF05}). \end{aligned}$$

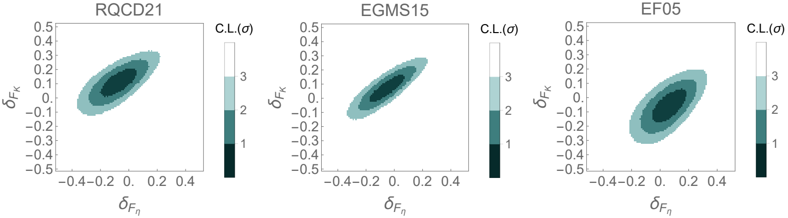

Figure: Constraints on higher order remainders, alternative inputs for F_η

Extraction of L_5^r

We may algebraically eliminate F_0 and L_4^r:

$$F_K^2 = \frac{F_\pi^2(1-\delta_{F_\pi}) + \frac{5}{2}\mu_\pi - \mu_K - \frac{3}{2}\mu_\eta + 4YM_\pi^2(\frac{3}{128\pi^2}\log\frac{\mu}{M_\rho} + L_5^r)(r-1)}{1-\delta_{F_K}}$$

$$F_\eta^2 = \frac{F_\pi^2(1-\delta_{F_\pi}) + 4\mu_\pi - 4\mu_K + \frac{8}{3}YM_\pi^2(\frac{3}{128\pi^2}\log\frac{\mu}{M_\rho} + L_5^r)(2r-2)}{1-\delta_{F_\eta}}$$

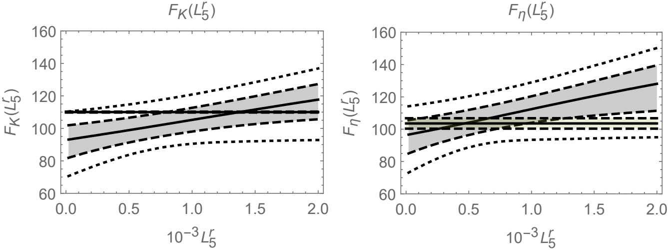

Figure: Theoretical predictions for F_K and F_η. 1σ and 2σ CL contours depicted. Horizontal - data [10, 3].

Extraction of L_5^r

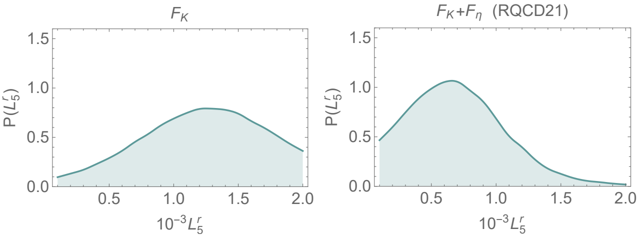

Figure: PDFs for L_5^r from F_K and F_η for $F_\eta = (1.123 \pm 0.035)F_\pi$ (RQCD21).

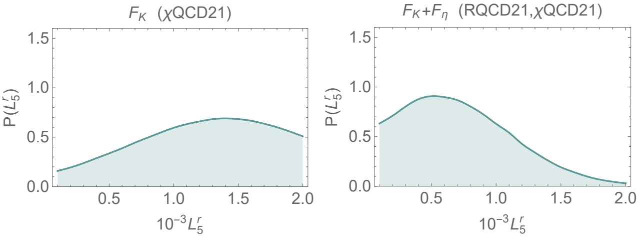

Figure: PDFs for L_5^r from F_K and F_η (RQCD21), (χQCD21).

We may eliminate L_5^r, to obtain the following results:

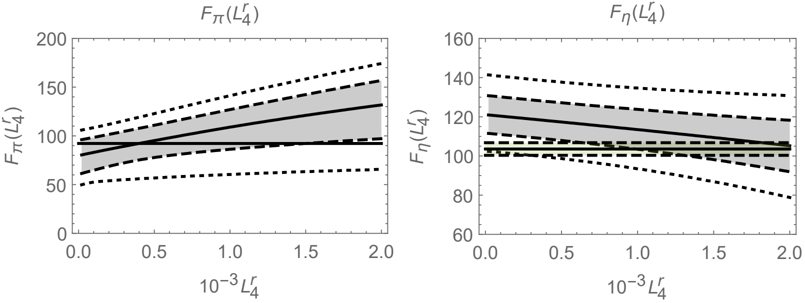

Figure: Theoretical predictions for F_π and F_η. 1σ and 2σ CL contours depicted. Horizontal - data from [10, 3].

Extraction of L_4^r

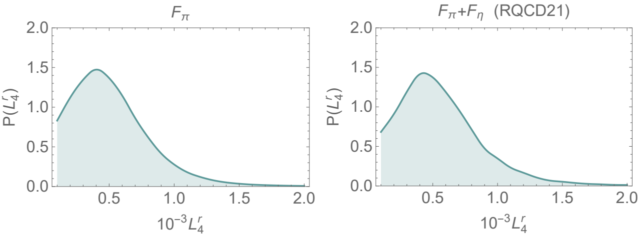

Figure: PDFs for L_4^r from F_π and F_η for $F_\eta = (1.123 \pm 0.035)F_\pi$ (RQCD21).

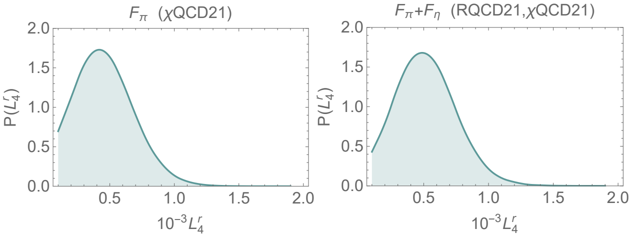

Figure: PDFs for L_4^r from F_π and F_η (RQCD21), using (χQCD21).

Final results

Theoretical prediction for the $SU(3)$ η meson decay constant

$$F_\eta = 117.5 \pm 9.4 \text{ MeV} = (1.28 \pm 0.10)F_\pi.$$

Utilizing data from lattice RQCD21[3], we obtained our main result

$$L_4^r = (0.44 \pm 0.37) \cdot 10^{-3} \qquad \text{(RQCD21)}$$

$$L_5^r = (0.66 \pm 0.37) \cdot 10^{-3} \qquad \text{(RQCD21)}$$

$$\delta_{F_K} = 0.10 \pm 0.07, \qquad \delta_{F_\eta} = -0.08 \pm 0.08, \qquad \rho = 0.71 \qquad \text{(RQCD21)}$$

We have also used recent computations of the LECs by χQCD21 [9]

$$L_4^r = (0.46 \pm 0.24) \cdot 10^{-3} \qquad (\text{RQCD21}, \chi\text{QCD21})$$

$$L_5^r = (0.68 \pm 0.42) \cdot 10^{-3} \qquad (\text{RQCD21}, \chi\text{QCD21})$$

	RQCD21	HPQCD 13A [11]	BE14 [12]	FF14 [12]	MILC10 [13]
$10^3 L_4^r$	0.44 ± 0.37	0.09 ± 0.34	$\equiv 0.3$	0.76 ± 0.18	0.02 ± 0.56
$10^3 L_5^r$	0.66 ± 0.37	1.19 ± 0.25	1.01 ± 0.06	0.50 ± 0.07	0.95 ± 0.41

Table: Our main result in comparison with literature

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