Chiral extrapolation of $\pi\pi$ scattering amplitudes and hadronic vacuum polarization

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11th Workshop on Chiral Dynamics, Bochum, August 29th, 2024

Niehus, Hoferichter, Kubis, Ruiz de Elvira, Phys. Rev. Lett. 126 (2021) 102002

Colangelo, Hoferichter, Kubis, Niehus, Ruiz de Elvira, PLB 825 (2022) 136852

QCD spectrum in the lattice

- important progress in understanding the QCD spectrum from first principles in lattice QCD
- computations at physical M_π available [Alexandrou et al. (2024), Boyle et al. (2023, 2024), Fischer et al. (2021), . . .]
- but most calculations still at unphysically large pion masses

  $\hookrightarrow$ extrapolation to the physical point required
- controlled using effective field theories: Chiral Perturbation Theory (ChPT)

  $\hookrightarrow$ limited to low-energies for perturbative observables

$$t(s)\big|_{\mathrm{ChPT}} = t_2(s) + t_4(s) + t_6(s) + \cdots$$

- ChPT satisfies unitarity only perturbative

  $\hookrightarrow$ resonance description requires unitarization
- Inverse amplitude method

$$t^{IAM}_{\mathrm{NLO}}(s) = \frac{t_2(s)^2}{t_2(s) - t_4(s)}, \quad t^{IAM}_{\mathrm{NNLO}}(s) = \frac{t_2(s)^2}{t_2(s) - t_4(s) - t_6(s) + t_4(s)^2/t_2(s)}$$

Two-loop IAM analysis

- IAM relies on the chiral perturbative expansion in M_π

  $\hookrightarrow$ applied to study **lattice data** and **resonance properties** for unphysical pion masses

  [Pelaez, Hanhart, Rios (2008), Pelaez, Nebreda (2010), Bolton, Briceño, Wilson (2016), Hu, Guo, Molina, Döring, Alexandru, Mai (2016, 2017, 2018)]

- but so far mostly applied at one-loop order

  $\hookrightarrow$ study the convergence of the IAM in M_π requires two-loop order

- two-loop analysis missing because

  $\triangleright$ no analytic two-loop expressions

  $\triangleright$ large number of N^2LO LECs leads to unstable fits

- in this talk for $\pi\pi$ scattering

  $\hookrightarrow$ analytic and compact expressions for two-loop amplitudes

  $\hookrightarrow$ strategy for a stable fit

  $\hookrightarrow$ application to the hadronic vacuum polarization

Analytic $\pi\pi$ partial waves for $J = 0, 1, 2$

- LO: current algebra results

$$t_0^0(s)\big|_2 = \frac{2s - M_\pi^2}{32\pi F^2}, \quad t_0^2(s)\big|_2 = -\frac{s - 2M_\pi^2}{32\pi F^2}, \quad t_1^1(s)\big|_2 = \frac{s - 4M_\pi^2}{96\pi F^2}, \quad t_2^I(s)\big|_2 = 0.$$

[Weinberg (1966)]

- NLO: the partial-wave amplitudes can be written in the form

$$\mathrm{Re}\, t_J^I(s)\big|_4 = \sum_{i=0}^{2} b_i^{IJ}(s)\,[L(s)]^i + \sum_{i=1}^{3} b_{l_i}^{IJ}(s)\, l_i^r, \quad L(s) = \log\frac{1+\sigma(s)}{1-\sigma(s)}, \quad \sigma(s) = \sqrt{1 - \frac{4M_\pi^2}{s}}$$

[Niehus, Hoferichter, Kubis, JRE (2021)]

  ▷ with $b_i^{IJ}(s)$ and $b_{l_i}^{IJ}(s)$ polynomials

- NNLO: expressions can be brought into similar form

$$\mathrm{Re}\, t_J^I(s)\big|_6 = \sum_{i=0}^{4} c_i^{IJ}(s)\,[L(s)]^i + \sum_{i=1}^{3} c_{l_i}^{IJ}(s)\, l_i^r + d^{II}(s)\Big[\sum_{n=\pm} \mathrm{Li}_3(\sigma_n(s)) - L(s)\,\mathrm{Li}_2(\sigma_-(s))\Big]$$
$$+ c_{l_3^2}^{IJ}(s)\big(l_3^r\big)^2 + P^{IJ}(s), \quad \sigma_\pm(s) = 2\sigma(s)/(\sigma(s)\pm 1) \quad \mathrm{Li}_i \equiv \text{polylogs.}$$

[Niehus, Hoferichter, Kubis, JRE (2021)]

  ▷ with $c_i^{IJ}(s)$, $c_{l_i}^{IJ}(s)$, $c_{l_3^2}^{IJ}(s)$ and $d^{II}(s)$ polynomials

  ▷ $P(s)$ polynomial containing N^{2}LO LECs r_i

$\pi\pi$ P-wave lattice data fit strategy

- work with pion decay constant in the chiral limit F
  - ▷ at LO: only F and M_π
  - ▷ at NLO: one LEC combination $l_2^r - l_1^r$
  - ▷ at N^{2}LO: three NLO LECs, l_1^r, l_2^r, l_3^r and three NNLO, r_a, r_b, r_c
- compute the IAM energy levels via Lüscher's quantization condition

$$\delta(E^*_{\pi\pi}) = \mathcal{Z}(E^*_{\pi\pi})$$

working in lattice units wherever possible

- for each lattice ensemble minimize

$$E_i^{\text{lat}} - E_i^{\text{IAM}}, \quad F_\pi^{\text{lat}} - F_\pi^{\text{ChPT}}$$

keeping lattice correlations

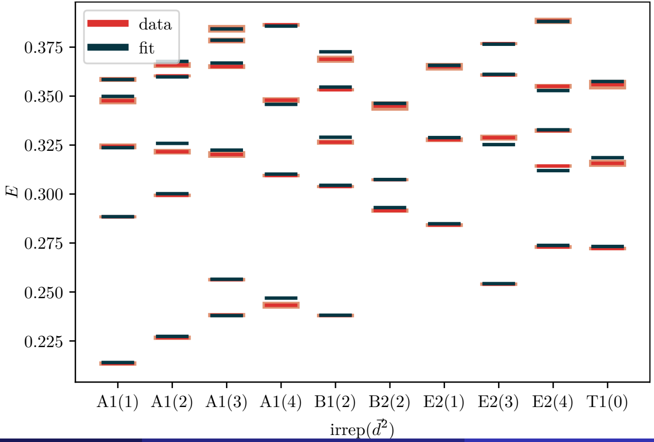
$\pi\pi$ P-wave fit to CLS data

- benchmark check: fit to 5 CLS ensembles [C. Andersen, J. Bulava, B. Hörz, and C. Morningstar (2019)]

  together with CLS data for F_π [M. Bruno, T. Korzec, and S. Schaefer (2017)]

  $\hookrightarrow$ 3 lattice spacings and $M_\pi \in [200, 283]$ MeV

- e.g. NNLO result for the D101 ensemble [Niehus, Hoferichter, Kubis, JRE (2021)]



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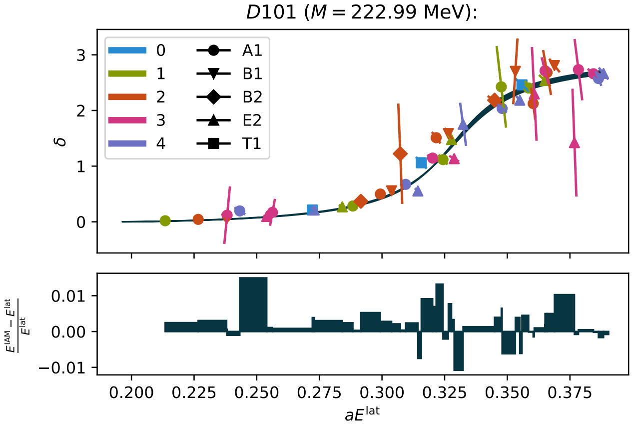

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- uncertainties:

  $\triangleright$ statistical errors: jackknife resampling

  $\triangleright$ lattice spacing: only enters in the renormalization scale μ

  $\triangleright$ truncation of the chiral expansion

  chiral expansion in $\alpha = M_\pi^2 / M_\rho^2$

  for a given chiral observable X, define the truncation error

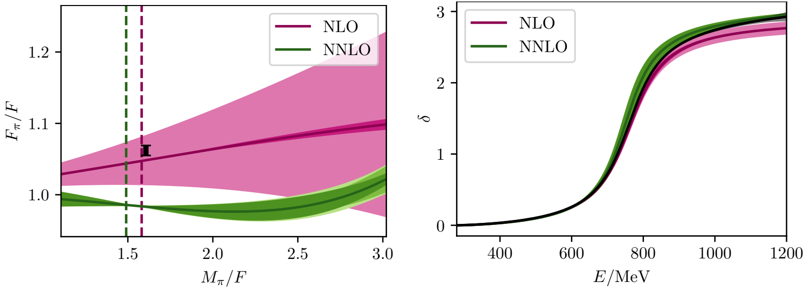
$$\Delta X_{\text{NLO}} = \alpha X_{\text{NLO}}, \quad \Delta X_{\text{NNLO}} = \max\left\{\alpha^2 X_{\text{NLO}}, \alpha \left|X_{\text{NLO}} - X_{\text{NNLO}}\right|\right\}$$

[Epelbaum, Krebs, Meißner (2015)]

$\pi\pi$ P-wave fit to CLS data: results

- fit results:

	NLO	NNLO
χ^2/dof	1.91	1.53
F/ MeV	88.27(0.23)(0.04)(2.86)	93.7(2.3)(0.1)(0.2)



- NNLO improvement but fit quality still not acceptable

[Niehus, Hoferichter, Kubis, JRE (2021)]

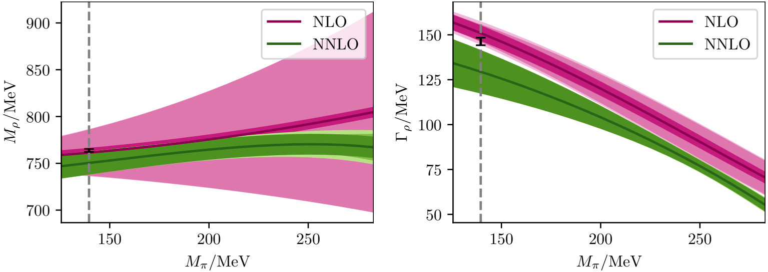
↪ truncation error dominates at NLO

- tension between F and ρ parameters at NNLO

↪ more detailed understanding of lattice artifacts required

$\pi\pi$ P-wave fit to CLS data: $\rho(770)$ results

	NLO	NNLO
M_ρ/MeV	761.4(5.1)(0.3)(24.7)	750(12)(1)(1)
Γ_ρ/MeV	150.9(4.4)(0.1)(4.9)	129(12)(1)(1)
Re g	5.994(54)(0)(194)	5.71(23)(2)(1)
$-$Im g	0.731(21)(0)(24)	0.46(14)(2)(1)



- compatible with phenomenological results [Niehus, Hoferichter, Kubis, JRE (2021)]

  small tension for the NNLO width
- RBC/UKQCD result: $M_\rho = 796(5)(50)$, $\Gamma_\rho = 192(10)(30)$ [Boyle et al. (2024)]

  $\hookrightarrow$ look at NNLO chiral extrapolation

Hadronic vacuum polarization from lattice QCD

- chiral extrapolation part of systematic error budget

  $\hookrightarrow$ extrapolation to (or interpolation around) physical quark masses

- biggest contribution from $I = 1$ ud isospin-symmetric correlator

  $\hookrightarrow$ phenomenologically dominated by 2π channel, first correction from 4π

- ChPT not enough

  Golterman, Maltman, Peris (2017)

$$a_\mu^{I=1} = \frac{\alpha}{24\pi^2}\left(-\log\frac{M_\pi^2}{m_\mu^2} - \frac{31}{6} + 3\pi^2\sqrt{\frac{M_\pi^2}{m_\mu^2}} + \mathcal{O}\left(\frac{M_\pi^2}{m_\mu^2}\log^2\frac{M_\pi^2}{m_\mu^2}\right)\right)$$

  $\hookrightarrow$ “convergence” in M_π/m_μ

- need to provide information on the $\rho(770)$ resonance

  $\hookrightarrow$ inverse amplitude method at two-loop order

Data-driven approach to HVP

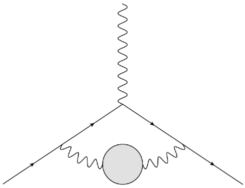
- data-driven approach the HVP

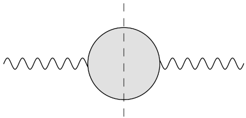
$$a_\mu^{\text{HVP}} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{s_{\text{thr}}}^{\infty} ds \frac{\hat{K}(s)}{s^2} R_{\text{had}}(s)$$

$\hookrightarrow$ expressed in terms of the R-ratio

$$R_{\text{had}}(s) = \frac{3s}{4\pi\alpha^2}\sigma(e^+e^- \to \text{hadrons})$$

[talk by Frederic Stieler]



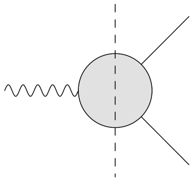


- two-pion contribution to $R_{\text{had}}(s)$

$$\sigma(e^+e^- \to \pi^+\pi^-) = \frac{\pi\alpha^2}{3s}\sigma_\pi^3(s)\left|F_\pi^V(s)\right|^2$$

$\hookrightarrow$ **pion vector form factor**

$$\langle\pi^\pm(p')|j_{\text{em}}^\mu(0)|\pi^\pm(p)\rangle = \pm(p'+p)^\mu F_\pi^V((p'-p)^2)$$



Dispersive representation of 2π contribution

- Decomposition of pion form factor

$$F_\pi^V(s) = \underbrace{\Omega_1^1(s)}_{\text{elastic } \pi\pi \text{ scattering}} \times \underbrace{G_\omega(s)}_{\text{isospin-breaking } 3\pi \text{ cut}} \times \underbrace{G_{\text{in}}(s)}_{\text{inelastic effects: } 4\pi,\ldots}$$

[talk by G. Colangelo]

- elastic $\pi\pi$ contribution via Omnès factor

$$\Omega_1^1(s) = \exp\left\{ \frac{s}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{\delta_1^1(s')}{s'(s'-s)} \right\}$$

- $G_\omega(s)$ does not contribute to $I=1$ correlator
- parameterized as normal or conformal polynomial

$\hookrightarrow$ free parameters can be matched to $\langle r_\pi^2 \rangle$

$$F_\pi^V(s)\big|_{I=1} = \left[1 + \left(\frac{\langle r_\pi^2\rangle}{6} - \dot{\Omega}_1^1(0)\right)s\right]\Omega_1^1(s)$$

Pion-mass dependence of 2π contribution

- Pion-mass dependence of $\langle r_\pi^2 \rangle$ at two loops known [Bijnens, Colangelo, Talavera (1998)]

  $\hookrightarrow$ at NNLO new LEC r^r_{V1}

- from resonance saturation $r^r_{V1} = 2.0 \times 10^{-5}$

  $\hookrightarrow$ in concord with lattice [Feng, Fu, Jin (2020)]

- LECs in $\delta^1_1(s)$: combined fit to

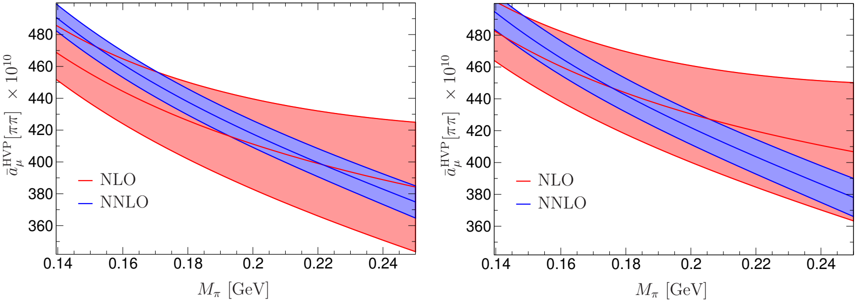
  $\triangleright$ CLS lattice data [C. Andersen, J. Bulava, B. Hörz, and C. Morningstar (2019)]

  $\triangleright$ dispersive $e^+e^- \to \pi^+\pi^-$ data analysis [Colangelo, Hoferichter, Stoffer (2019)]

  $\hookrightarrow$ describes the physical point within uncertainties

$$
\begin{aligned}
a_\mu^{\text{HVP}}[\pi\pi, \leq 1\,\text{GeV}]\big|_{I=1} &= 486.3(1.4)(2.1) \times 10^{-10} \\
a_\mu^{\text{HVP}}[\pi\pi, \leq 1\,\text{GeV}]\big|^{\text{NLO}}_{I=1} &= 460.4(0.3)(14.9)(7.2) \times 10^{-10}, \\
a_\mu^{\text{HVP}}[\pi\pi, \leq 1\,\text{GeV}]\big|^{\text{NNLO}}_{I=1} &= 482.4(0.1)(0.7)(8.0) \times 10^{-10},
\end{aligned}
$$

- pion mass dependence of $a_\mu^{\rm HVP}[\pi\pi, \leq 1\,\text{GeV}]$



[Colangelo, Hoferichter, Kubis, Niehus, JRE (2021)]

$\hookrightarrow$ stable results, pion-mass dependence of 2π under control

[Golterman, Maltman, Peris (2017)]

Possible application to lattice QCD

- chiral LECs as fit parameters:
  - ▷ describes $\pi\pi$ physics
  - ▷ need to add $a_\mu^{\text{HVP}}[ud, I = 1, \text{non} - \pi\pi] = \xi + M_\pi \psi$
  - ▷ can provide independent constraints from other lattice calculations: $\delta_1^1(s)$, F_π^V, r_{v1}^r
- simple parameterizations:
  - ▷ possible for space-like integrand

$$\frac{\bar{\Pi}(-Q^2)}{Q^2} = \frac{a + bQ^2}{1 + cQ^2 + dQ^4},$$

↪ test infrared singularities

  - ▷ fits to {a, b, c, d} indicate singularity as strong as M_π^{-2} in [0.14, 0.25] GeV

$$f_1(M_\pi^2) = \frac{x}{M_\pi^2} + y + zM_\pi^2, \quad f_2(M_\pi^2) = \frac{x}{M_\pi^2} + y \log M_\pi^2 + z$$

↪ could help inform lattice fits

Summary

- analytic and compact results for $\pi\pi$ partial waves at two-loop order

  ▷ available as Mathematica notebook in arXiv submission 2009.04479

  ↪ no need to stay at one-loop order anymore!

- new strategy for a stable NNLO IAM fit to lattice data

  ↪ study of IAM convergence in M_π

- good fit quality requires good understanding of lattice artifacts

- application to the 2π contribution to the HVP

  ↪ stable results

  ↪ strategy to control chiral extrapolation of HVP

Spare slides

The Inverse Amplitude Method

- dispersion relation for $G(s) = t_2(s)^2/t(s)$

$$G(s) = G(0) + sG'(0) + s^2 G''(0) + \frac{s^3}{\pi}\int_{s_{th}}^{\infty} ds' \frac{\text{Im } t(s')^{-1}}{s'^3(s'-s)} + \frac{s^3}{\pi}\int_{-\infty}^{0} ds' \frac{\text{Im } t(s')^{-1}}{s'^3(s'-s)}$$

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- unitarity in the physical region

$$\text{Im } t(s) = \sigma(s)|t(s)|^2 \quad \Rightarrow \quad \text{Im}\, t^{-1}(s) = -\sigma(s), \quad \Rightarrow \quad \text{Im}\, G(s) = -\text{Im}\, t_4(s)$$

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$$t^{IAM}_{\text{NLO}}(s) = \frac{t_2(s)^2}{t_2(s) - t_4(s)}$$

[Truong, Herrero, Dobado (1990)]

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- subtraction constants: can be evaluated in ChPT

$$G(0) = t_2(0)/t(0) \simeq t_2(0) - t_4(0), \quad G'(0) \simeq t_2(0)' - t_4(0)', \quad G''(0) \simeq -t_4(0)''$$

- LHC contribution: approximated in ChPT $\quad \Rightarrow \quad \text{Im } G(s) \sim \text{Im } t_2(s)^2/t(s)\Big|_{\text{ChPT}} = -\text{Im } t_4(s)$
- altogether:

$$t^{IAM}_{\text{NLO}}(s) = \frac{t_2(s)^2}{t_2(s) - t_4(s)}, \quad t^{IAM}_{\text{NNLO}}(s) = \frac{t_2(s)^2}{t_2(s) - t_4(s) - t_6(s) + t_4(s)^2/t_2(s)}$$

[Truong, Herrero, Dobado (1990)]

The Inverse Amplitude Method

- dispersion relation for $G(s) = t_2(s)^2/t(s)$

$$G(s) = t_2(s) - t_4(s)$$

- unitarity in the physical region

$$\text{Im}\ t(s) = \sigma(s)|t(s)|^2 \quad \Rightarrow \quad \text{Im}\, t^{-1}(s) = -\sigma(s), \quad \Rightarrow \quad \text{Im}\, G(s) = -\text{Im}\, t_4(s)$$

- subtraction constants: can be evaluated in ChPT

$$G(0) = t_2(0)/t(0) \simeq t_2(0) - t_4(0), \quad G'(0) \simeq t_2(0)' - t_4(0)', \quad G''(0) \simeq -t_4(0)''$$

- LHC contribution: approximated in ChPT $\Rightarrow$ $\text{Im}\ G(s) \sim \text{Im}\ t_2(s)^2/t(s)\Big|_{\text{ChPT}} = -\text{Im}\, t_4(s)$
- altogether:

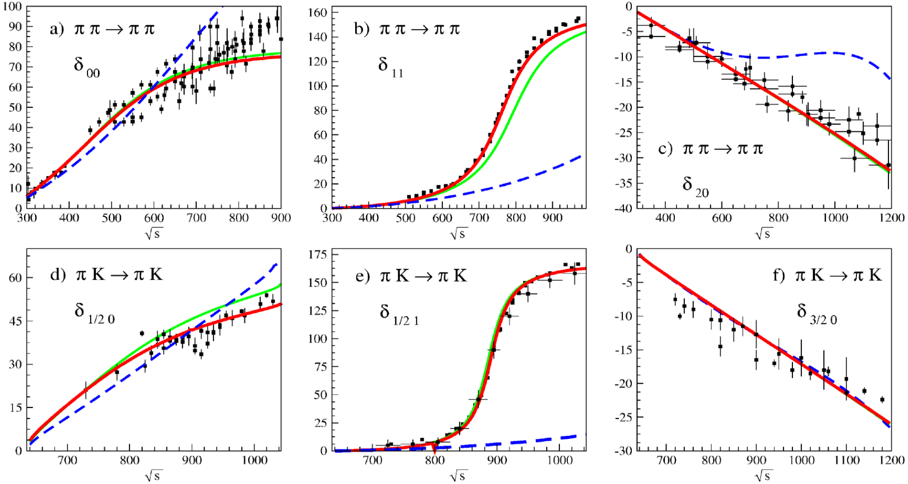
$$t^{IAM}_{\text{NLO}}(s) = \frac{t_2(s)^2}{t_2(s) - t_4(s)}, \quad t^{IAM}_{\text{NNLO}}(s) = \frac{t_2(s)^2}{t_2(s) - t_4(s) - t_6(s) + t_4(s)^2/t_2(s)}$$

[Truong, Herrero, Dobado (1990)]

▷ satisfies exact unitarity + chiral low-energy expansion

▷ derived from a **dispersion relation**

The IAM and scattering data

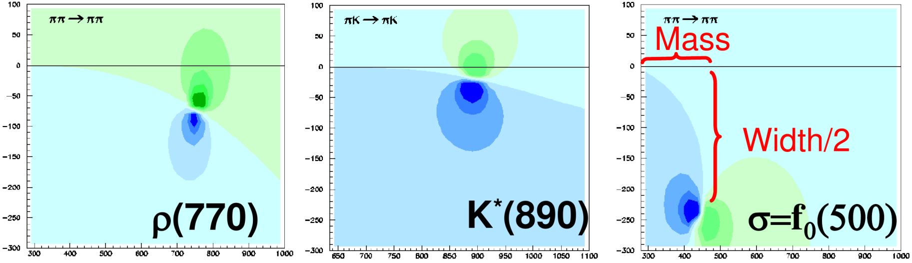


[Gómez-Nicola,Pelaez (2002)]

- perturbative ChPT, IAM fit 1, IAM fit 2

- **derived** from a **dispersion relation**

  ↪ **analytic continuation** to the complex plane



- similar for the $f_0(980)$, $\kappa(700)$, $a_0(980)$

Possible application to lattice QCD