Short-distance contributions to Hadronic-light-by-light for the muon $g-2$

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Vetenskapsrådet

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Introduction

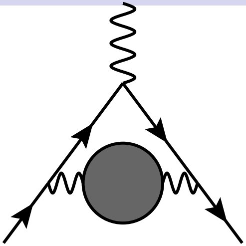
Other talks related to the muon anomalous magnetic moment

- Gilberto Colangelo: Dispersive approach to hadronic contributions to the Muon $g-2$
- Hartmut Wittig: The puzzles of the muon anomalous magnetic moment
- Simon Holz: Dispersive determination of the eta/eta' transition form factors
- And many more related ones

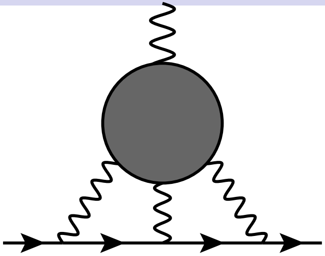
Experimental measurement:

- define the anomaly: $a_\mu = \dfrac{g_\mu - 2}{2}$
- $a_\mu = 0.00116592059(22)$ or $1.9 \cdot 10^{-7}$ (0.19 ppm)
- $g_\mu = 2.00233184118(44)$ or $2.2 \cdot 10^{-10}$ (experimentalists are (too?) modest)
- Can we calculate this to the same precision?
- All theory uncertainties under sufficient control except for the hadronic contributions

Hadronic contributions



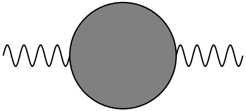
LO-HVP



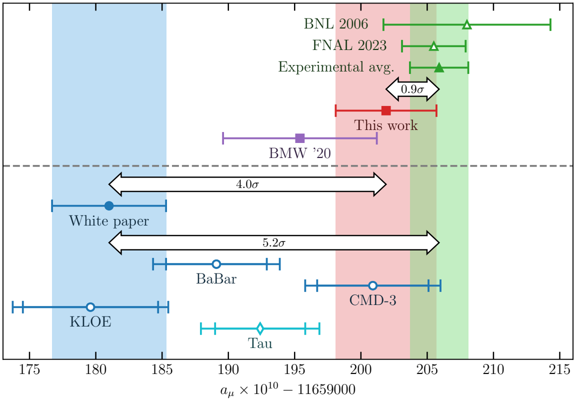
HLbL

- Muon and photon lines, representative diagrams
- The blobs are hadronic contributions
- There are higher order contributions of both types: known accurately enough
- $a_\mu^{HVP} = 6845(40)\ 10^{-11}$ (LO+NLO+NNLO)(White paper; error has increased)
- $a_\mu^{HLbL} = 92(18)\ 10^{-11}$ (LO+NLO)(White paper)
- $a_\mu^{exp} - a_\mu^{QED} - a_\mu^{EW} = 7186(22) \cdot 10^{-11}$
- Difference: $\boxed{\Delta a_\mu = 249(49) \cdot 10^{-11}}$

HVP

-  = Two-point function of two electro-magnetic currents Π
- Integrate over a weight function
- Can do that in:
  - Minkowski momentum space (dispersive approach)
  - Euclidean momentum space (early lattice QCD and MUonE)
  - Euclidean space (in principle lattice QCD)
  - Time-momentum representation (mixed; present lattice QCD)
- These are all related due to the analyticity property of two-point functions
- Simple: only one variable
- Problem: need 0.3% precision to match experimental a_μ

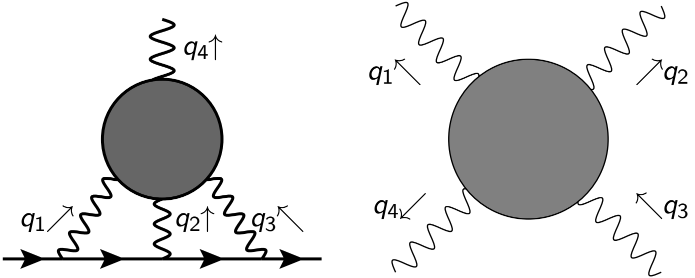
HVP: status



Source: BMW24, arXiv:2407.10913

- All other numbers taken from white-paper HLBL = 92(18)
- Details: talks this morning
- I will not further comment on this but lots of ongoing work

HLbL: Hadronic light-by-light



- $=\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3)$ of four vector currents (not two)
- 6 variables (not just one)
- Actually we really need $\left.\frac{\delta\Pi^{\mu\nu\lambda\sigma}(q_1,q_2,q_3)}{\delta q_{4\rho}}\right|_{q_4=0}$
- Mixed: q_4 at zero, q_1^2, q_2^2, q_3^2 so three-variables, or Q_1^2, Q_2^2, Q_3^2 $(q_i^2 = -Q_i^2)$
- Models, Dispersive methods, Lattice QCD

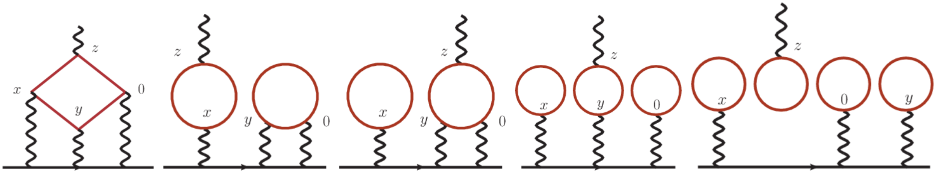
HLbL dispersive history

- late 1990s: two groups (Kinoshita, Bijnens); models and physics sense: 83(32) (BPP) after counting proposed by de Rafael
- Lots of work on the single pion exchange 2000-2015 (Knecht, Nyffeler,...)
- Start of connection with QCD (Melnikov, Vainshtein 2003)
- Always a problem of separating contributions
- Breakthough in 2015: how to do dispersive consistently (Colangelo,...)
- Also connection to short-distance major progress (Bijnens, Hermansson-Truedsson, Rodriguez-Sanchez)
- Main remaining: 3 pion and medium mass resonances: much work in progress
- Comment: numbers consistent over many years but errors on much better footing now

Contributions HLbL White paper

- "Long distance": under good control
  - Dispersive method: Berne group around G. Colangelo
  - π^0 (and η, η') pole: $93.8(4.0) \cdot 10^{-11}$ (BPP 85(13))
  - Pion and kaon box (pure): $-16.4(2) \cdot 10^{-11}$
  - $\pi\pi$-rescattering (include scalars below 1 GeV): $-8(1) \cdot 10^{-11}$
- Charm (beauty, top) loop: $3(1) \cdot 10^{-11}$
- "Short and medium distance" Main source of the error
  - Scalars, tensors: $-1(3) \cdot 10^{-11}$
  - Axial vector: $6(6) \cdot 10^{-11}$
  - Short-distance: $15(10) \cdot 10^{-11}$
- $a_\mu^{HLbL} = 92(19) \cdot 10^{-11}$
- Since then:
  - Short distance constraints improved (this talk)
  - Axial vectors better understood
  - Work in progress to put all together better

HLbL Lattice QCD



Eur.Phys.J.C 81 (2021) 651

- Three independent groups (similar methods), latest results
- RBC/UKQCD 23 $124.7(14.9) \cdot 10^{-11}$
- Mainz 21/22 $109.6(15.9) \cdot 10^{-11}$
- BMW preliminary $126.8(13) \cdot 10^{-11}$
- Dispersive $92(19) \cdot 10^{-11}$
- Other lattice methods: calculate formfactors needed in the dispersive method $\pi^0, \eta \to \gamma^* \gamma^*$

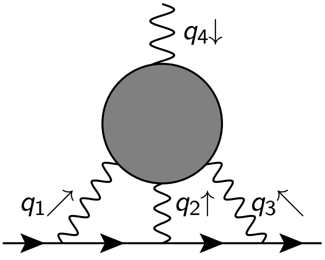
What are we up against?

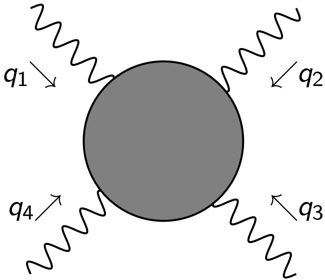
- Lots of resonances in light meson table from PDG (2022) 1-1.5 GeV

$\phi(1020)$	$h_1(1170)$	$b_1(1235)$	$a_1(1260)$	$f_2(1270)$
$f_1(1285)$	$\eta(1295)$	$\pi(1300)$	$a_2(1320)$	$f_0(1370)$
$\pi_1(1400)$	$\eta(1405)$	$h_1(1415)$	$f_1(1420)$	$\omega(1420)$
$a_0(1450)$	$\rho(1450)$	$\eta(1475)$	$f_0(1500)$	...

- couplings to on-shell photons known for very few
- off-shell photons ($q_i^2 \neq 0$) even less
- Clearly we will need to go beyond data as it is now
- More data will always be useful as a constraint and we will still need improvement around 1 GeV

Definitions





$= \Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3)$

- Actually we really need $\left.\frac{\delta \Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3)}{\delta q_{4\rho}}\right|_{q_4=0}$
- Never purely short-distance: q_4 at zero
- $q_i^2 = -Q_i^2$

Definitions

$$\Pi^{\mu\nu\lambda\sigma} = -i \int d^4x d^4y d^4z e^{-i(q_1\cdot x + q_2\cdot y + q_3\cdot z)} \left\langle T\left(j^\mu(x) j^\nu(y) j^\lambda(z) j^\sigma(0)\right)\right\rangle$$

Use the Colangelo et al. 2017 conventions (mainly)

$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma}\hat{\Pi}_i, \qquad \left.\frac{\delta\Pi^{\mu\nu\lambda\sigma}}{\delta q_{4\rho}}\right|_{q_4=0} = \sum_{i=1}^{54} \left.\frac{\delta T_i^{\mu\nu\lambda\sigma}}{\delta q_{4\rho}}\hat{\Pi}_i\right|_{q_4=0} = \sum_{i=1}^{19} P_i^{\mu\nu\lambda\sigma\rho}\tilde{\Pi}_i$$

$$a_\mu = \frac{2\alpha^3}{3\pi^2}\int_0^\infty dQ_1 dQ_2 Q_1^3 Q_2^3 \int_{-1}^{1} d\tau\sqrt{1-\tau^2}\sum_{i=1}^{12}\hat{T}_i\left(Q_1, Q_2, \tau\right)\overline{\Pi}_i\left(Q_1, Q_2, \tau\right)$$

$$Q_3^2 = Q_1^2 + Q_2^2 + 2Q_1Q_2\tau$$

- The 12 $\overline{\Pi}_i$ from $\hat{\Pi}_i$ for $i = 1, 4, 7, 17, 39, 54$
- These can in turn be derived from five of the $\tilde{\Pi}_i$
- But beware of keeping the permutations when doing approximations

Short-distance constraints

- There are very many different types of short-distance constraints (SDC)
- Those on hadronic properties
  - Couplings of hadrons to off-shell photons
  - Pure OPE (e.g. $\pi^0 \to \gamma^*\gamma^*$ at $Q_1^2 = Q_2^2$)
  - Brodsky-Lepage-Radyushkin-$\cdots$:
    - the overall power is very well predicted (counting rules)
    - the coefficient follows from the asymptotic wave functions and possible α_S corrections: larger uncertainty
  - Light-cone QCD sum rules
  - $\cdots$
- This type is mainly used in HLbL to put constraints on the form-factors in the individual contributions

Short-distance constraints

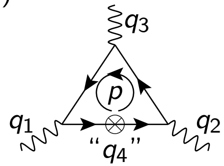
- On the full four-point function (4, 3 or 2 currents close)
- SD4: $\Pi^{\mu\nu\lambda\sigma}(q_1,q_2,q_3)$ all $Q_i \cdot Q_j$ large: the standard OPE
- SD3: $\left.\frac{\delta\Pi^{\mu\nu\lambda\sigma}(q_1,q_2,q_3)}{\delta q_{4\rho}}\right|_{q_4=0}$ with $Q_1^2 \sim Q_2^2 \sim Q_3^2 \gg \Lambda_{QCD}^2$

  JB,LL,NHT,ARS 19-21
- SD2: $\left.\frac{\delta\Pi^{\mu\nu\lambda\sigma}(q_1,q_2,q_3)}{\delta q_{4\rho}}\right|_{q_4=0}$ and $Q_1^2 \sim Q_2^2 \gg Q_3^2 (\gg \Lambda_{QCD}^2)$

  Melnikov-Vainshtein 03, JB, NHT, ARS 21-23
- Collaborators: Nils Hermansson-Truedsson, Laetitia Laub, Antonio Rodríguez-Sánchez
- SD3:
  - Phys.Lett. B798 (2019) 134994 [arxiv:1908.03331]: principle, next nonperturbative term
  - JHEP 10 (2020) 203 [arxiv:2008.13487]: proper description and up to NNLO nonperturbative terms
  - JHEP 04 (2021) 240 [arxiv:2101.09169]: perturbative correction
- SD2: JHEP 02 (2023) 167 [arxiv:2211.17183]: OPE in MV at $\alpha_S = 0$

Short-distance: correctly

- Similar problem in QCD sum rules for electromagnetic radii and magnetic moments
- Ioffe, Smilga, Balitsky, Yung, 1983
- For the q_4-leg use a constant background field and do the OPE in the presence of that constant background field
- Use radial gauge: $A_4^\lambda(w) = \frac{1}{2} w_\mu F^{\mu\lambda}$
  whole calculation is immediately with $q_4 = 0$.
- First term is exactly the massless quark loop
  (quark masses: next order)

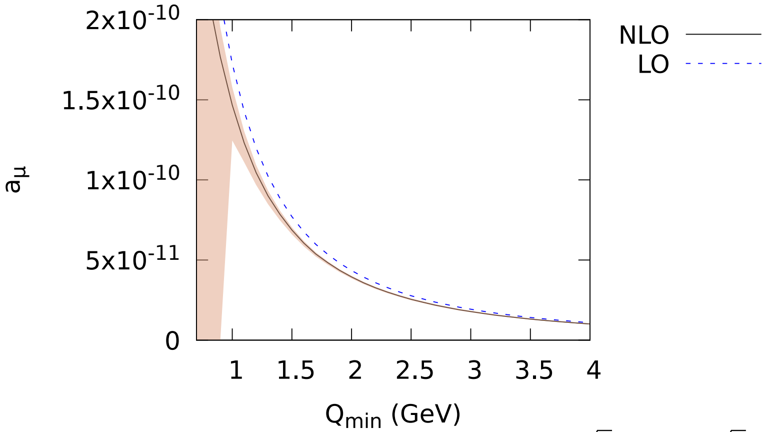


- 3 quark currents close

Results for SD3

- First part along in white paper Phys.Lett. B798 (2019) 134994 [arxiv:1908.03331]
- SD3 work reported in previous Chiral Dynamics meeting
- Leading term is the naive massless quark-loop
  (not true for quark mass corrections)
- Known fully analyticallly for pure quark loop and gluonic corrections
  JHEP 10 (2020) 203 [arxiv:2008.13487], JHEP 04 (2021) 240 [arxiv:2101.09169]
- Higher order terms in the OPE known and are small for $Q_i \geq 1$ GeV JHEP 10 (2020) 203 [arxiv:2008.13487]

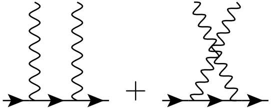
Perturbative corrections: numerics



- Uncertainty estimated by $\alpha_S(\mu)$ with $Q_{\text{min}}/\sqrt{2} \le \mu \le \sqrt{2}Q_{\text{min}}$
- Running $\alpha_S(M_Z)$ at 5 loops to $\alpha_S(m_\tau)$ or $\alpha_S(\mu)$
- Perturbative corrections are under control and negative, about -10%

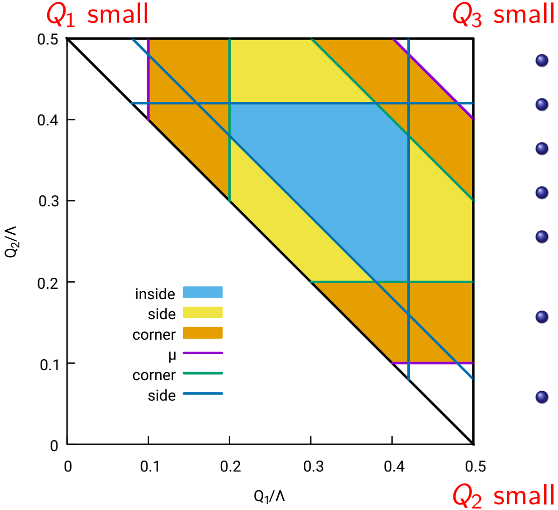
SD2 or MV short-distance

- K. Melnikov, A. Vainshtein, Phys. Rev. **D70** (2004) 113006. [hep-ph/0312226]
- take $Q_1^2 \approx Q_2^2 \gg Q_3^2$: Leading term in OPE of two vector currents is proportional to axial current
- $\Pi^{\rho\nu\alpha\beta} \propto \frac{P^\rho}{Q_1^2}\langle 0|\, T\left(J_A^\nu J_V^\alpha J_V^\beta\right)|0\rangle$ J_A comes from



- Coefficient of J_A has α_S and higher order OPE corrections
- AVV triangle anomaly: in particular nonrenormalization theorems
    - fully for longitudinal ($\overline{\Pi}_i$, $i = 1, 2, 3$)
    - perturbative for the others
- Recent discusions, implementations,...: M. Knecht, JHEP 08 (2020) 056 [2005.09929], P. Masjuan, P. Roig and P. Sanchez-Puertas, J. Phys. G **49** (2022) no.1, 015002 [2005.11761] Colangelo et al, JHEP 03 (2020) 101 [1910.13432], Eur.Phys.J.C 81 (2021) 8, 702 [2106.13222], Melnikov and Vainshtein, [1911.05874], L. Cappiello et al., Phys. Rev. D **102** (2020) no.1, 016009 [1912.02779], J. Leutgeb and A. Rebhan, Phys. Rev. D **104** (2021) 094017 [2108.12345] J. Lüdtke and M. Procura, Eur. Phys. J. C **80** (2020) no.12, 1108 [2006.00007],...

SD2, MV and corner



- SD2
- MV
- Corner
- All refer to the same kinematics
- Corner comes from the triangle at fixed $Q_1 + Q_2 + Q_3 = \Lambda$
- There might be regions where both *SD*2 and *SD*3 are applicable
- Corners:
  Orange=perturbative
  White=nonperturbative

MV Short-distance: known before

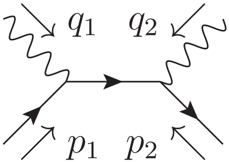
- Before only a proper prediction for $\hat{\Pi}_1$ Colangelo et al, JHEP 03 (2020) 101 [1910.13432], Eur.Phys.J.C 81 (2021) 8, 702 [2106.13222]
- $\overline{Q}_3 = Q_1 + Q_2$, $Q_3 \ll Q_1, Q_2$
- $\hat{\Pi}_1 = \frac{e_q^4}{\pi^2} \frac{-12}{Q_3^2 \overline{Q}_3^2} \left(1 - \frac{\alpha_S}{\pi}\right)$
- The quark loop and its gluonic correction reproduce this
- JB,NHT,ARS,JHEP 02 (2023) 167 [arxiv:2211.17183] and in progress: calculate the corrections: gluonic and OPE
- Next term in OPE has a number of features (from our corner expansions of the masless case):
  - $\log \frac{Q_3^2}{\overline{Q}_3^2}$ show up already at $\alpha_S = 0$ (now understood in the OPE picture)
  - For some of the terms the gluonic corrections dominate
- Should allow to resum some of the large corrections at the corners

Starting point

- We define:
$$\Pi^{\mu_1\mu_2} = \frac{i}{e^2}\int d^4x_1 \int d^4x_2 e^{-i(q_1x_1+q_2x_2)}\langle 0|T(J^{\mu_1}(x_1)J^{\mu_2}(x_2))|\gamma(q_3)\gamma(q_4)\rangle$$
- allows to get the four point function needed for a_μ: $\Pi^{\mu_1\mu_2} = \epsilon_{\mu_3}\epsilon_{\nu_4}\Pi^{\mu_1\mu_2\mu_3\nu_4}$. and the needed $\partial/\partial q_{4,\mu_4}$ at $q_4 \to 0$ as well
- OPE on the two currents: work out
$$\int d^4x_1 \int d^4x_2 e^{-i(q_1x_1+q_2x_2)} T(J^{\mu_1}(x_1)J^{\mu_2}(x_2))$$
for $\hat{q} = (q_1 - q_2)/2$ with $\hat{Q}^2 = -\hat{q}$ large, $q_3 = -q_1 - q_2$ is small.

First result

- Look at the diagram and add $q_1 \leftrightarrow q_2$:
- expanding in $\hat{q}$ gives



$$\Pi^{\mu_1\mu_2}_{\bar{q}q} \approx -\frac{e_q^2}{e^2}\frac{-\hat{q}_\alpha}{\hat{q}^2}\langle 0|\bar{q}(0)[\gamma^{\mu_1}\gamma^{\alpha}\gamma^{\mu_2} - \gamma^{\mu_2}\gamma^{\alpha}\gamma^{\mu_1}]q(0))|\gamma(q_3)\gamma(q_4)\rangle$$

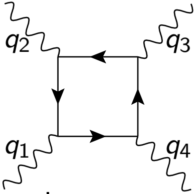
$$-\frac{ie_q^2}{e^2\hat{q}^2}(g_{\mu_1\delta}g_{\mu_2\beta} + g_{\mu_2\delta}g_{\mu_1\beta} - g_{\mu_1\mu_2}g_{\delta\beta})$$

$$\times\left(g^{\alpha\delta} - 2\frac{\hat{q}^\delta\hat{q}^\alpha}{\hat{q}^2}\right)\langle 0|\bar{q}(0)(\overrightarrow{D}^\alpha - \overleftarrow{D}^\alpha)\gamma^\beta q(0))|\gamma(q_3)\gamma(q_4)\rangle$$

- First line $D = 3$, next lines line $D = 4$
- Need more $D = 4$ terms: various combinations of $F_{\mu\nu}F_{\alpha\beta}$ and $G_{\mu\nu}G_{\alpha\beta}$

Where do these come from?

- Comes from limit $q_3, q_4 \to 0$ using background gauge of:



and the same diagram with the low-energy legs replaced by gluons

- Gives:

$$\begin{aligned}
O^{\mu_1\mu_2} =& \frac{8}{\hat{q}^2} F^{\mu_1\gamma} F^{\mu_2\delta} \hat{q}_\gamma \hat{q}_\delta - \frac{16}{3\hat{q}^6} \hat{q}^{\mu_1} \hat{q}^{\mu_2} F^{\alpha\gamma} F_{\alpha\delta} \hat{q}_\gamma \hat{q}^\delta \\
&+ \left(-\frac{32}{3} + \frac{16}{3} B(\hat{q}^2)\right) \left(\frac{1}{\hat{q}^2} F^{\mu_1\alpha} F^{\mu_2}{}_\alpha + \frac{1}{\hat{q}^4} F^{\mu_1\alpha} F_{\alpha\beta} \hat{q}^{\mu_2} \hat{q}^\beta + \frac{1}{\hat{q}^4} F^{\mu_2\alpha} F_{\alpha\beta} \hat{q}^{\mu_1} \hat{q}^\beta\right) \\
&+ \left(-\frac{8}{3} + \frac{8}{3} B(\hat{q}^2)\right) \left(\frac{2}{\hat{q}^4} F^{\alpha\gamma} F_{\alpha\delta} \hat{q}^\delta \hat{q}_\gamma g^{\mu_1\mu_2} + \frac{1}{\hat{q}^4} F^{\alpha\beta} F_{\alpha\beta} \hat{q}^{\mu_1} \hat{q}^{\mu_2} - \frac{1}{\hat{q}^2} F^{\alpha\beta} F_{\alpha\beta} g^{\mu_1\mu_2}\right)
\end{aligned}$$

and a similar expression with gluon field strengths

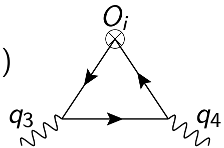
- with $B(\hat{q}^2) = \frac{2}{\epsilon} + 2 - \log(\hat{q}^2)$

Gluonic corrections

- $\lim_{q_4\to 0} \frac{\partial}{\partial q_{4,\mu 4}} \langle 0|O_i|\gamma(q_3)\gamma(q_4)\rangle$
- Calculation at $\mu = \hat{Q}$
- Gluonic corrections add one more operator (and changes coefficients of the others)
- $D = 3$: $\hat{q}_\alpha \bar{q}(0)[\gamma^{\mu_1}\gamma^\alpha\gamma^{\mu_2} - \gamma^{\mu_2}\gamma^\alpha\gamma^{\mu_1}]q(0)$
- $D = 4$:
  - $O_1^{\alpha\beta} = \bar{q}(0)(\overrightarrow{D}^\alpha - \overleftarrow{D}^\alpha)\gamma^\beta q(0))$
  - $O_2 = F^{\alpha\gamma}F^\beta{}_\gamma$
  - $O_3 = F^{\gamma\delta}F_{\gamma\delta}g^{\alpha\beta}$
  - $O_4 = G^{\alpha\gamma}G^\beta{}_\gamma$
  - $O_5 = G^{\gamma\delta}G_{\gamma\delta}g^{\alpha\beta}$
  - $O_6 = \bar{q}(0)\left(\gamma^\alpha\gamma^\beta\gamma^\gamma + \gamma^\gamma\gamma^\beta\gamma^\alpha\right)\left(\overrightarrow{D}^\gamma + \overleftarrow{D}^\gamma\right) q(0)$
- All (Wilson) coefficients known analytically and function of $\log(\hat{q}^2/\mu^2)$
- O_6 can be removed using equations of motion in terms of O_1

Next step (in progress)

- Use RGE to run down to $\mu = Q_3$
- O_1 and $F_{\mu\nu}F_{\alpha\beta}$ mix via $(q_3, q_4 \to 0)$



- Take matrix elements from the $D = 3$ and $D = 4$ quark/gluon operators to e^2 and $F_{\mu\nu}F_{\alpha\beta}$ to e^0 (now not $q_3 \to 0$ as in the step at $\mu = \hat{Q}$) at the low scale
- For the moment we calculate the matrix elements at $\mu = \hat{Q}$
- The full result is of course μ independent

Matrix elements

- We now have $\lim_{q_4\to 0}(\partial/\partial q_{4,\mu_4})\Pi^{\mu_1\mu_2\mu_3\nu_4}$ to two powers in $\hat{q}$
- Note gauge invariance for q_3 exact
- $q_4 \to 0$ gauge invariance is antisymmetry in $\nu_4\mu_4$
- BUT gauge invariance for q_1, q_2 only perturbatively in $\hat{q}$
- Consequence: be careful when using gauge equivalent expressions
- In particular when using projectors to get quantities without Lorentz indices (our intermediate $\widetilde{\Pi}$ or the $\hat{\Pi}_i$) need to use the projectors with lowest powers of $\hat{q}$ possible

Perturbative matrix elements

- $D = 3$ reproduce that only in $\hat{\Pi}_1$ in the corner with Q_3 small there is a contribution (not in Q_1 or Q_2 small)
- $D = 4$:
  - Matrix elements of the operators still contain kinematical singularities:

    (Q_3 small; $\delta_{12} = Q_1 - Q_2$ small): $$\hat{\Pi}_7(O_1) = \frac{32\delta_{12} Q_3^2}{3\pi^2 (Q_3^2 - \delta_{21}^2) \overline{Q}_3^5}$$
  - Similar for other $\hat{\Pi}_i$ and
  - When projecting on $\hat{\Pi}_i$: need to be very careful with powers in $\hat{q}$
  - Gauge invariance in q_3 fully correct
  - Gauge invariance in q_1, q_2 only correct perturbative in $1/\hat{q}$
  - Use projectors with as low powers of $\hat{q}$ as possible
- $D = 3$ and $D = 4$ mix because of $\hat{q}$ powers in the projectors
- In the end: UV and kinematic singularities cancel when all contributions are added also including gluonic corrections

Results

$\overline{Q}_3 = Q_1 + Q_2$, $\overline{Q}_2 = Q_1 + Q_3$

Corner with q_3 small:

$$\hat{\Pi}_1 = -\frac{4}{\pi^2\, Q_3^2\, \overline{Q}_3^2} + \mathcal{O}\left(\overline{Q}_3^{-4}\right)$$

$$\hat{\Pi}_4 = -\frac{16}{3\pi^2\, \overline{Q}_3^4} + \mathcal{O}\left(\overline{Q}_3^{-5}\right)$$

$$\hat{\Pi}_7 = \mathcal{O}\left(\overline{Q}_3^{-6}\right)$$

$$\hat{\Pi}_{17} = \frac{16}{3\pi^2\, Q_3^2\, \overline{Q}_3^4} + \mathcal{O}\left(\overline{Q}_3^{-5}\right)$$

$$\hat{\Pi}_{39} = \frac{16}{3\pi^2\, Q_3^2\, \overline{Q}_3^4} + \mathcal{O}\left(\overline{Q}_3^{-5}\right)$$

$$\hat{\Pi}_{54} = \mathcal{O}\left(\overline{Q}_3^{-5}\right)$$

Corner with q_2 small

$$\hat{\Pi}_1 = -\frac{16\left(5 + 6\log 2\,\frac{Q_2}{\overline{Q}_2}\right)}{9\pi^2\, \overline{Q}_2^4} + \mathcal{O}\left(\overline{Q}_2^{-5}\right)$$

$$\hat{\Pi}_4 = -\frac{4}{3\pi^2\, Q_2^2\, \overline{Q}_2^2} + \mathcal{O}\left(\overline{Q}_2^{-3}\right)$$

$$\hat{\Pi}_7 = -\frac{16}{3\pi^2\, Q_2^2\, \overline{Q}_2^4} + \mathcal{O}\left(\overline{Q}_2^{-5}\right)$$

$$\hat{\Pi}_{17} = \mathcal{O}\left(\overline{Q}_2^{-5}\right)$$

$$\hat{\Pi}_{39} = \frac{16}{3\pi^2\, Q_2^2\, \overline{Q}_2^4} + \mathcal{O}\left(\overline{Q}_2^{-5}\right)$$

$$\hat{\Pi}_{54} = -\frac{8}{3\pi^2\, Q_2^2\, \overline{Q}_2^4} + \mathcal{O}\left(\overline{Q}_2^{-5}\right)$$

Results perturbative matrix elements

$\overline{Q}_1 = Q_2 + Q_3$

Corner with q_1 small

$$\hat{\Pi}_1 = -\frac{16\left(5 + 6\log 2\,\frac{Q_1}{\overline{Q}_1}\right)}{9\pi^2\,\overline{Q}_1^4} + \mathcal{O}\left(\overline{Q}_1^{-5}\right)$$

$$\hat{\Pi}_4 = -\frac{4}{3\pi^2\,Q_1^2\,\overline{Q}_1^2} + \mathcal{O}\left(\overline{Q}_1^{-3}\right)$$

$$\hat{\Pi}_7 = \mathcal{O}\left(\overline{Q}_1^{-4}\right)$$

$$\hat{\Pi}_{17} = \mathcal{O}\left(\overline{Q}_1^{-5}\right)$$

$$\hat{\Pi}_{39} = \frac{16}{3\pi^2\,Q_1^2\,\overline{Q}_1^4} + \mathcal{O}\left(\overline{Q}_1^{-5}\right)$$

$$\hat{\Pi}_{54} = \frac{8}{3\pi^2\,Q_1^2\,\overline{Q}_1^4} + \mathcal{O}\left(\overline{Q}_1^{-5}\right)$$

- Agrees with quark loop expansion
- Gluon corrections are known (not RGE yet) and agree with the expansion form previous work
- Higher orders can depend on $\delta_{23} = Q_2 - Q_3, \ldots$

Nonperturbative matrix elements

- Look at more general Q_3, no longer $Q_3^2 \gg \Lambda_{QCD}^2$
- We cannot directly calculate all required matrix elements
- We know they can only depend on q_3
- Parametrize the matrix elements in the most general way
- $D = 3$ operator has two form-factors $w_L(q_3^2)$ and $q_T(q_3^2)$

$$\lim_{q_4 \to 0} \frac{\partial \Pi^{\mu_1\mu_2\mu_3\nu_4}}{\partial q_{4,\,\mu_4}} = \frac{1}{2\pi^2}\,\frac{q_3^2}{\hat{q}^2}\,\epsilon^{\mu_1\mu_2\hat{q}\delta}\Big(\epsilon_{\mu_3\mu_4\nu_4\delta}\,\omega_T(q_3^2) - \frac{1}{q_3^2}\,\epsilon_{q_3\mu_4\nu_4\delta}\,q_{3\mu_3}\,\omega_T(q_3^2)$$

$$+ \frac{1}{q_3^2}\,\epsilon_{\mu_3\mu_4\nu_4 q_3}\,q_{3\delta}\,\left[\omega_L(q_3^2) - \omega_T(q_3^2)\right]\Big)\,.$$

- Full agreement with MV (and the other discussions)
- Leading contribution: (only for q_3 small)

$$\hat{\Pi}_1 = \frac{2}{\pi^2 \overline{Q}_3^2}\,\omega_L(q_3^2)$$

Nonperturbative matrix elements

- The $D = 3$ operator gives contributions at the next order in $1/\hat{q}$

q_3 small:

$$\hat{\Pi}_{17} = -\frac{4Q_3^2}{\pi^2(Q_3^2-\delta_{12}^2)\overline{Q}_3^4}\,\omega_T(q_3^2)$$

$$\hat{\Pi}_{39} = -\frac{4Q_3^2}{\pi^2(Q_3^2-\delta_{12}^2)\overline{Q}_3^4}\,\omega_T(q_3^2)$$

q_1 small:

$$\hat{\Pi}_{4} = \frac{Q_1^2}{\pi^2(Q_1^2-\delta_{23}^2)\overline{Q}_1^2}\,\omega_T(q_1^2)$$

$$\hat{\Pi}_{7} = -\frac{4Q_1^2\delta_{23}}{\pi^2(Q_1^2-\delta_{23}^2)\overline{Q}_1^3}\,\omega_T(q_1^2)$$

$$\hat{\Pi}_{39} = -\frac{4Q_1^2}{\pi^2(Q_1^2-\delta_{23}^2)\overline{Q}_1^4}\,\omega_T(q_1^2)$$

$$\hat{\Pi}_{54} = -\frac{4Q_1^2}{\pi^2(Q_1^2-\delta_{23}^2)\overline{Q}_1^4}\,\omega_T(q_1^2)$$

q_2 small:

$$\hat{\Pi}_{4} = \frac{Q_2^2}{\pi^2(Q_2^2-\delta_{31}^2)\overline{Q}_2^2}\,\omega_T(q_2^2)$$

$$\hat{\Pi}_{7} = \frac{4Q_2^2}{\pi^2(Q_2^2-\delta_{31}^2)\overline{Q}_2^4}\,\omega_T(q_2^2)$$

$$\hat{\Pi}_{39} = -\frac{4Q_2^2}{\pi^2(Q_2^2-\delta_{31}^2)\overline{Q}_2^4}\,\omega_T(q_2^2)$$

$$\hat{\Pi}_{54} = \frac{4Q_2^2}{\pi^2(Q_2^2-\delta_{31}^2)\overline{Q}_2^4}\,\omega_T(q_2^2)$$

Nonperturbative matrix elements

- The $D = 3$ operator gives contributions at the next order as well
- These have kinematical singularities
- The $D = 4$ operators contribute at this level as well
- The sum of all contributions cannot have kinematical singularities
- Leads to relations of the matrix elements between different operators

Nonperturbative matrix elements

- $O_1^{\alpha\beta} = \bar{q}(0)(\overrightarrow{D}^\alpha - \overleftarrow{D}^\alpha)\gamma^\beta q(0)$
- $\frac{\partial}{\partial q_{4\nu_4}} \langle 0 | O_1^{\alpha\beta} | \gamma^{\mu_3}(q_3)\gamma^{\mu_4}(q_4)\rangle\Big|_{q_4=0}$ has six form-factors
- We find three relations for these

$$\omega_{(8)}^{D,2} = -2\,\omega_{(8)}^{D,1} + \omega_{(8)}^{D,5} - \frac{\omega_{(8)}^{D,6}}{2} - \frac{\omega_{T,(8)}\,Q_i^2}{8\pi^2}\,,$$

$$\omega_{(8)}^{D,3} = -2\,\omega_{(8)}^{D,1} + \omega_{(8)}^{D,5} - \frac{\omega_{(8)}^{D,6}}{2} + \frac{\omega_{T,(8)}\,Q_i^2}{8\pi^2}\,,$$

$$\omega_{(8)}^{D,4} = \omega_{(8)}^{D,5}$$

- Models for the form-factors should satisfy these
- Using these get expressions for the $\hat{\Pi}_i$ in terms of all the form-factors
  Can be found in JHEP 02 (2023) 167 [arxiv:2211.17183]

Nonperturbative matrix elements

Work in progress:

- Phenomenological estimates of these form-factors
- Get a numerical estimate of the $D = 4$ contributions
- Complete the RGE running
- Put all together for an updated short-distance contribution to HLbL

Conclusions

- SD3 or all Q_i large
  - Massless quark loop is first term of a proper OPE expansion for HLbL
  - We have shown how to properly go to higher orders
  - We have calculated the next two terms in the OPE
    Numerically not relevant at the present precision
  - Gluonic corrections about -10%
- SD2 or MV or two Q_i large and the third much smaller
  - We have shown how to do an OPE and worked out the first two terms
  - Leading term full agreement with MV
  - Gluonic corrections calculated
  - Perturbative matrix elements: agreement with limits from SD3
  - Kinematic singularities require relations between form-factors of NLO and LO operators
- Why do this:
  - use QCD to identify poissibly large missing parts
  - match sum hadronic contributions to QCD at short distances
  - Finding the onset of the asymptotic domain