Left-hand branch cuts in lattice QCD scattering calculations

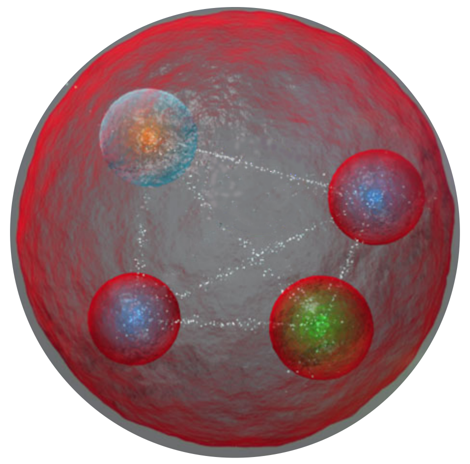
Maxwell T. Hansen

August 27th, 2024

THE UNIVERSITY of EDINBURGH

Motivation: Multi-hadron observables

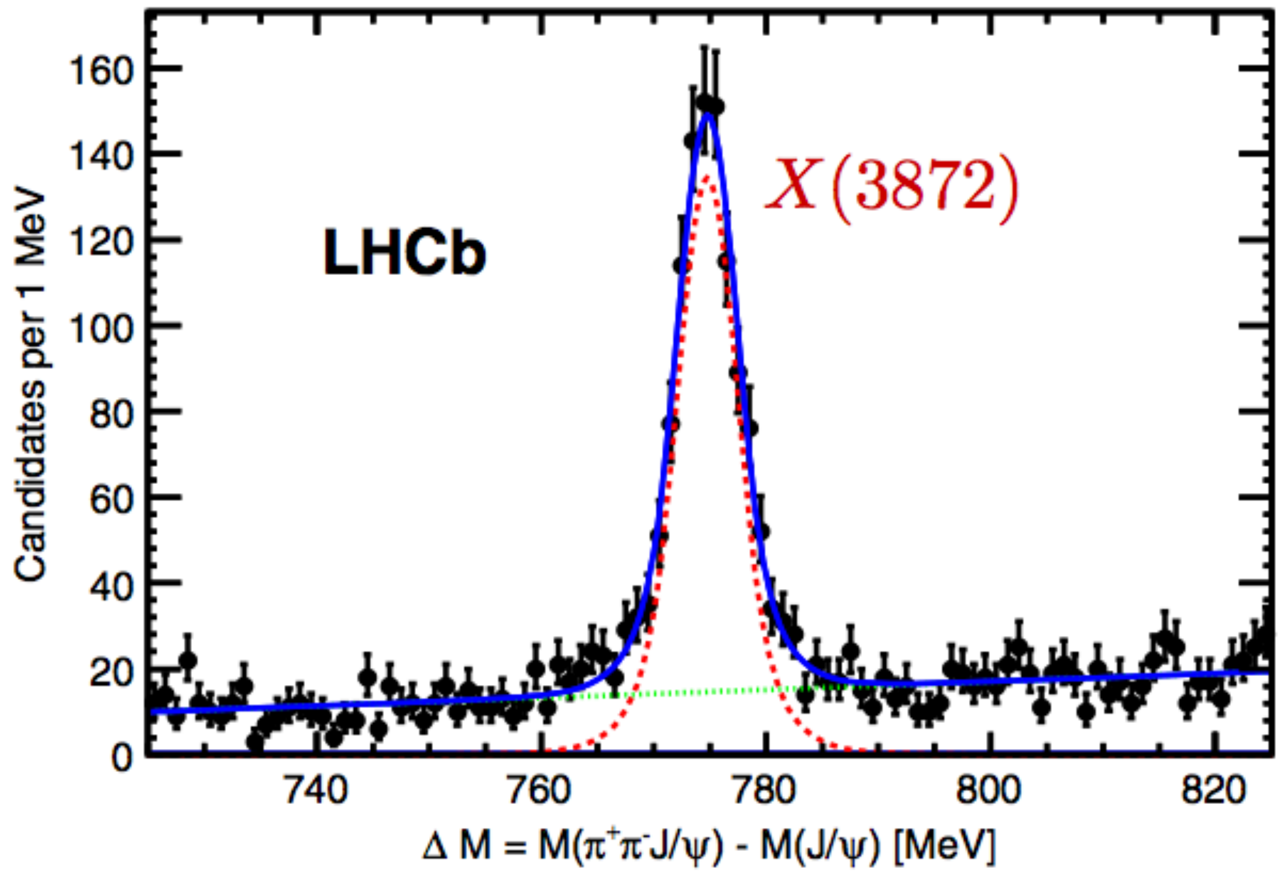
- Exotics, XYZs, tetra- and penta-quarks, *H* dibaryon



e.g. X(3872)

$$\sim |D^0\overline{D}^{*0} + \overline{D}^0 D^{*0}\rangle\,?$$

- LHCb (PRD92, 2015)



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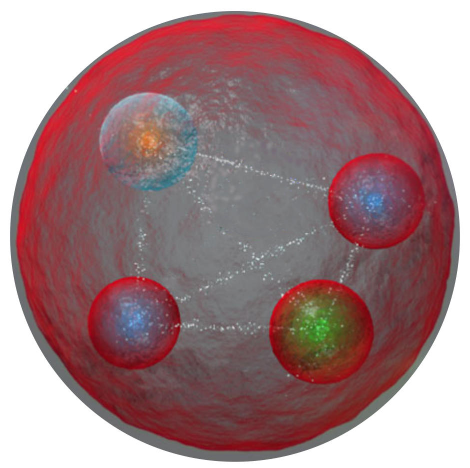

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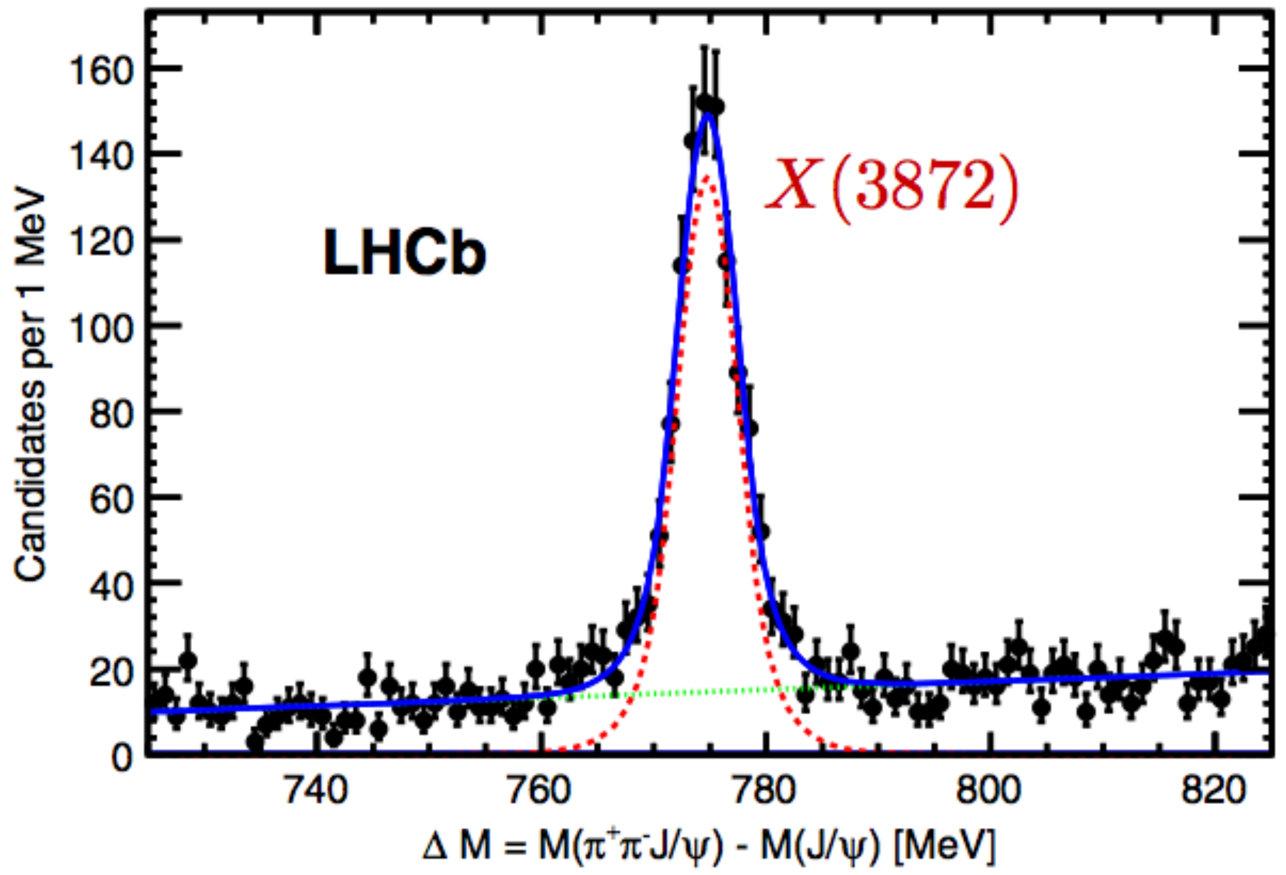

- Eletroweak, CP violation, resonant enhancement

CP violation in charm $D \to \pi\pi, K\overline{K}$

$$\Delta A_{CP} = (-15.4 \pm 2.9) \times 10^{-4}$$

• LHCb (PRL, 2019) •

$f_0(1710)$ could enhance ΔA_{CP}

• Soni (2017) •

Resonant B decays $B \to K^*\,\ell\ell \to K\pi\,\ell\ell$

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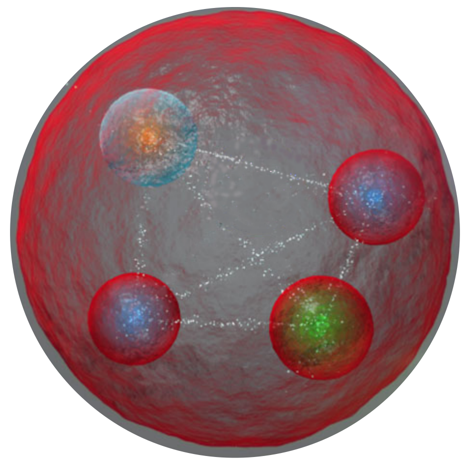

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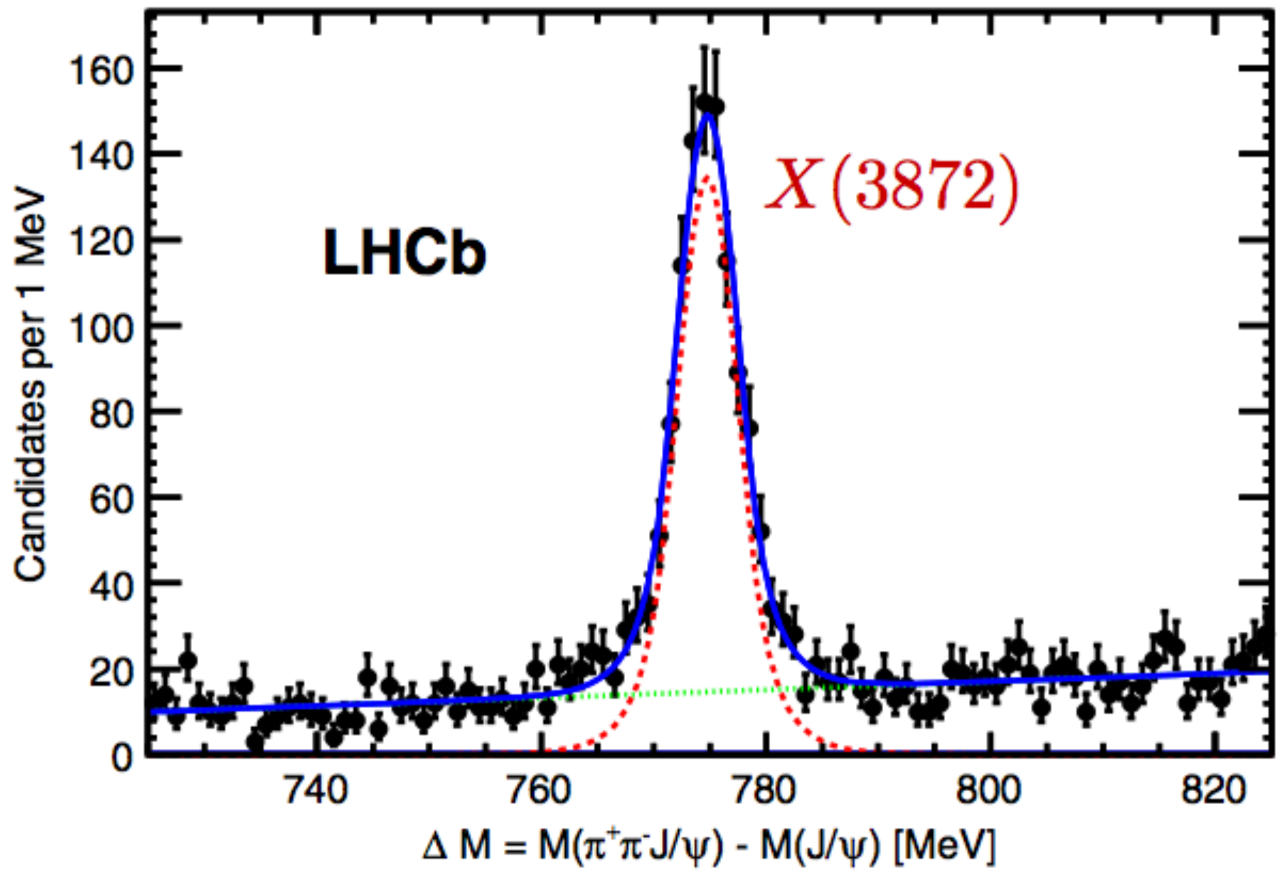

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Resonant B decays $B \to K^*\,\ell\ell \to K\pi\,\ell\ell$

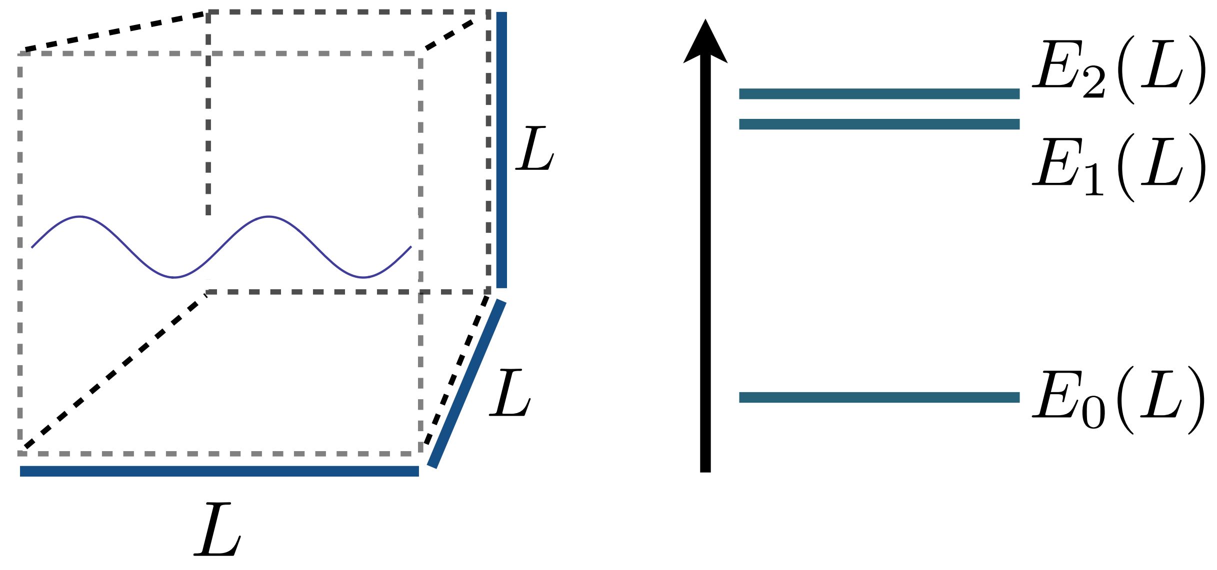
$|X\rangle, |\rho\rangle, |K^*\rangle, |f_0\rangle \notin$ **QCD Fock space**

Importance of the finite volume

$$|X\rangle, |\rho\rangle\,, |K^*\rangle\,, |f_0\rangle \quad \notin$$ **QCD Fock space**

$$|\pi\pi, \mathrm{out}\rangle, \quad |K\pi, \mathrm{out}\rangle, \quad \cdots \in$$ **QCD Fock space (continuum of states)**

Relation is (highly) non-trivial



$\in$ **Discrete set of finite-volume states**

Talks at this meeting

Nucleon resonances from lattice QCD
Colin Morningstar
Monday, 10:45 - 11:25

Three-hadron dynamics from lattice QCD
Fernando Romero-Lopez
Tuesday, 9:40 - 10:20

Interactions between two hadrons in lattice QCD
Sinya Aoki
10:45 - 11:25

Status of two-baryon scattering in lattice QCD
Jeremy Green
11:25 - 12:05

Recent Applications of Nuclear Lattice Effective Theory
Dean Lee
12:05 - 12:45

Monday afternoon (titles abbreviated)...

Scattering of Goldstone bosons on the lattice	*Ferenc Pittler*
Role of branchcuts for the Tcc(3875)	*Meng-Lin Du*
Charmonium resonances in coupled-channel scattering	*David Wilson*
Internal structure of the Tcc(3875)	*Michael Abolnikov*
ChPT and lattice studies of doubly charmed baryons	*De-Liang Yao*

Tuesday afternoon (titles abbreviated)...

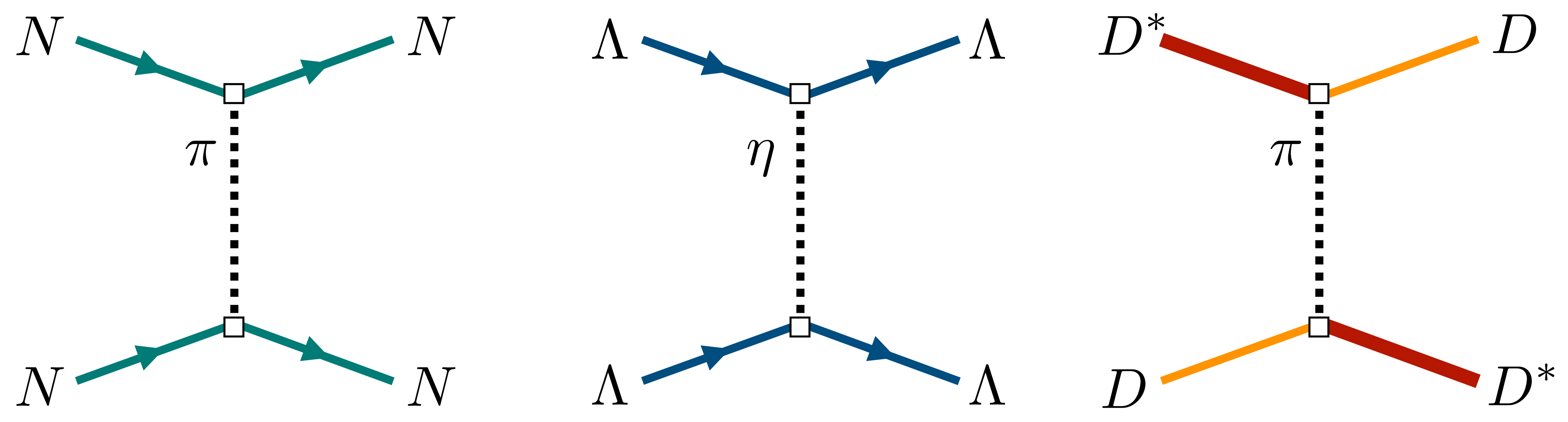
Six-meson scattering and three pions on the lattice	*Mattias Sjö*
Long-range forces in a finite volume	*Akaki Ruseksky*
Left-hand cut problem in lattice QCD	*Lu Meng*
Novel method for resonance positions in finite volume	*Congwu Wang*
Three-body analysis of Tcc(3875)	*Sebastian Dawid*
Three-meson scattering with physical quarks	*Fernando Romero-Lopez*

Thursday afternoon (titles abbreviated)...

Lattice calculation of $K \to \pi\pi$ decay $\pi\pi$ and scattering	*Masaaki Tomii*

Motivation: Left hand cuts

- Lattice QCD has seen significant recent progress in calculations of
  - baryon-baryon scattering (NPLQCD, CalLatt, Mainz)
  - vector-pseudoscalar scattering, e.g. $DD^* \to DD^*$ (relevant for the T_{cc})
- These amplitudes have sub-threshold branch cuts (from light meson exchanges)



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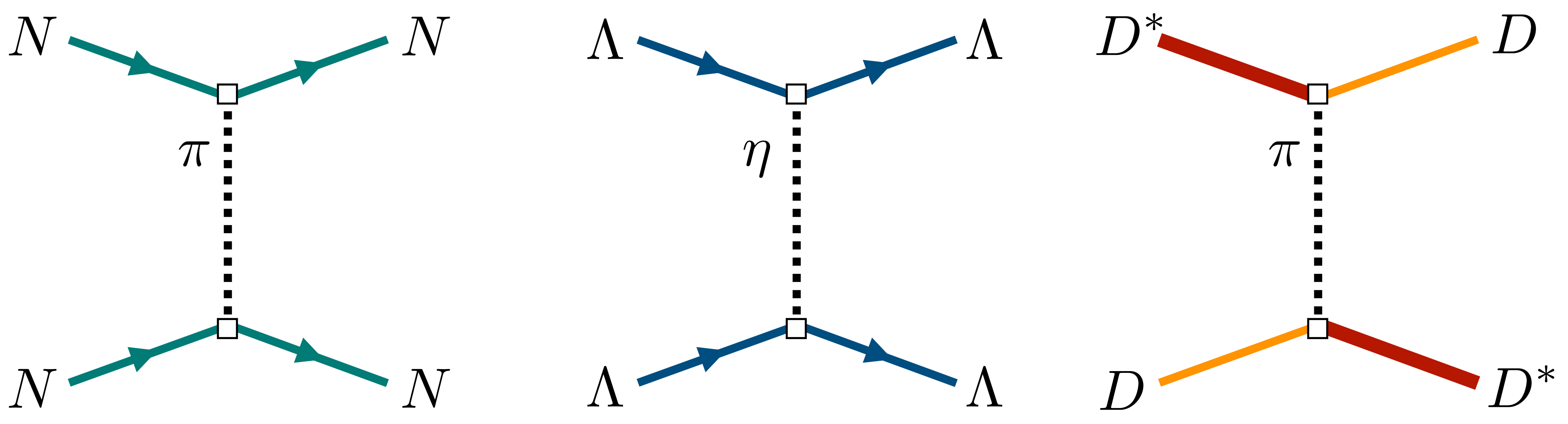

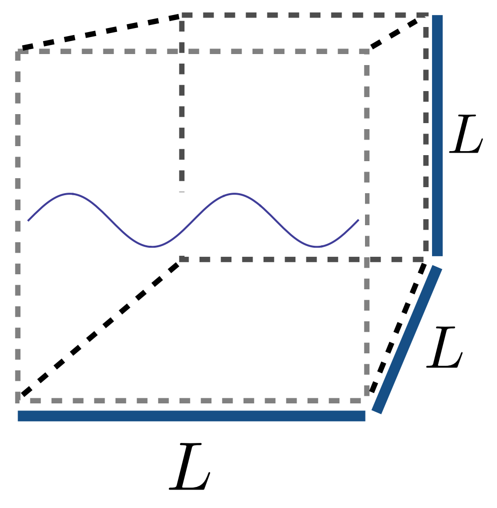

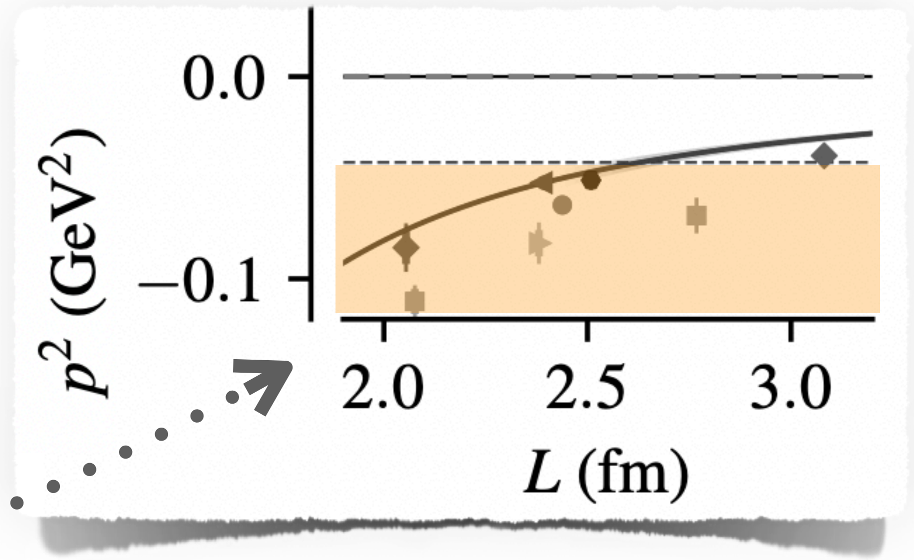

Green, et. al., PRL (2021)

- Calculations often extract finite-volume energies on the cuts
- The Lüscher scattering formalism (+extensions) is not applicable for these energies

$$\mathrm{Im}\left[p\cot\delta(p)_{\text{correct}} \Big|_{\text{cut}} \right] \neq 0 \qquad \mathrm{Im}\left[p\cot\delta(p)_{\text{from}} \Big|_{\text{cut}} \right] = 0$$

Goal: Provide a generalized quantization condition that resolves the issue

Related work…

- Early studies of long-range effects of NN scattering in a finite volume
  - $\exp[-\mu L]$ effects can be large due to [long-range effects] = [single-pion exchange] = [left-hand cut]

Sato and Bedaque, PRD (2007)

- Plane-wave basis to treat long-range interactions
  - avoids angular-momentum projection

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Meng et al., PRD (2024)

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- Generalisation of the Lüscher + K-matrix workflow (this talk)

MTH and Raposo, JHEP (2024)

- Alternative modification of the Lüscher formula via "modified effective range expansion"

Bubna et al., JHEP (2024)

- Three-body framework (automatically includes left-hand cut)

Dawid, Islam, Briceño, PRD (2023)

MTH, Romero-López and Sharpe, PRD (2024)

Additional related work…

- EFT context for long-range forces

  Modified effective range function, van Haeringen and Kok, Phys.Rev.A 1982

  Removing pions from two nucleon effective field theory, Steele and Furnstahl, Nucl.Phys.A 1999

  Coulomb effects in low-energy proton proton scattering, Kong and Ravndal, Nucl.Phys.A 2000

- Finite-volume QED context for long-range forces

  Two-Particle Elastic Scattering in a Finite Volume Including QED, Beane and Savage, PRD 2014

- Tcc context

  Coupled-channel approach to Tcc+ including three-body effects, Du et al., PRD 2022

  Role of Left-Hand Cut Contributions on Pole Extractions from Lattice Data: Case Study for Tcc(3875), Du et al., PRL 2023

- Numerical lattice QCD calculations

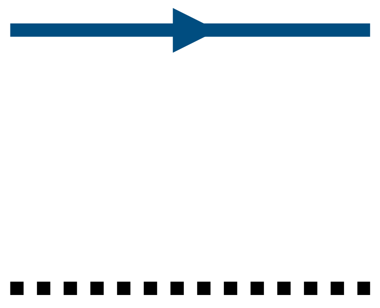
  Weakly bound H dibaryon from SU(3)-flavor-symmetric QCD, Green et al., PRL 2021

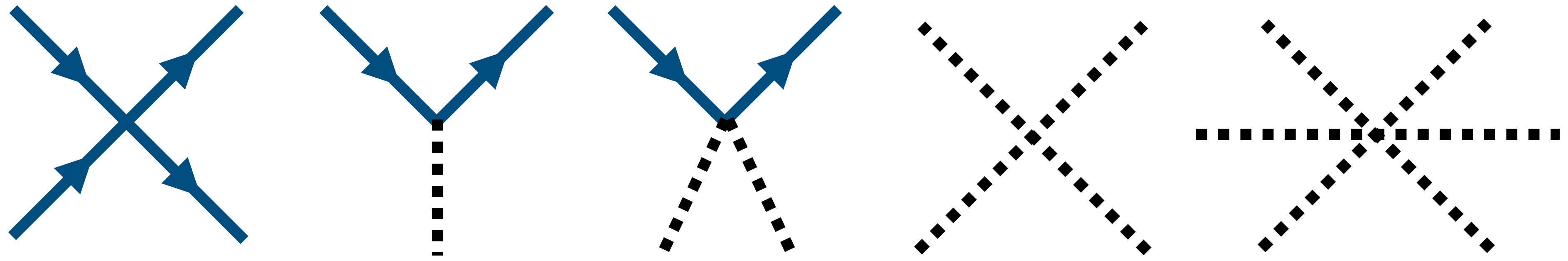
  Signature of a Doubly Charm Tetraquark Pole in DD Scattering on the Lattice*, Padmanath and Prelovsek, PRL 2022

  Near-threshold states in coupled DD∗−D∗D∗ scattering from lattice QCD Whyte, Wilson, and Thomas, arXiv:2405.15741 [hep-lat]

Introduction to the left-hand cut

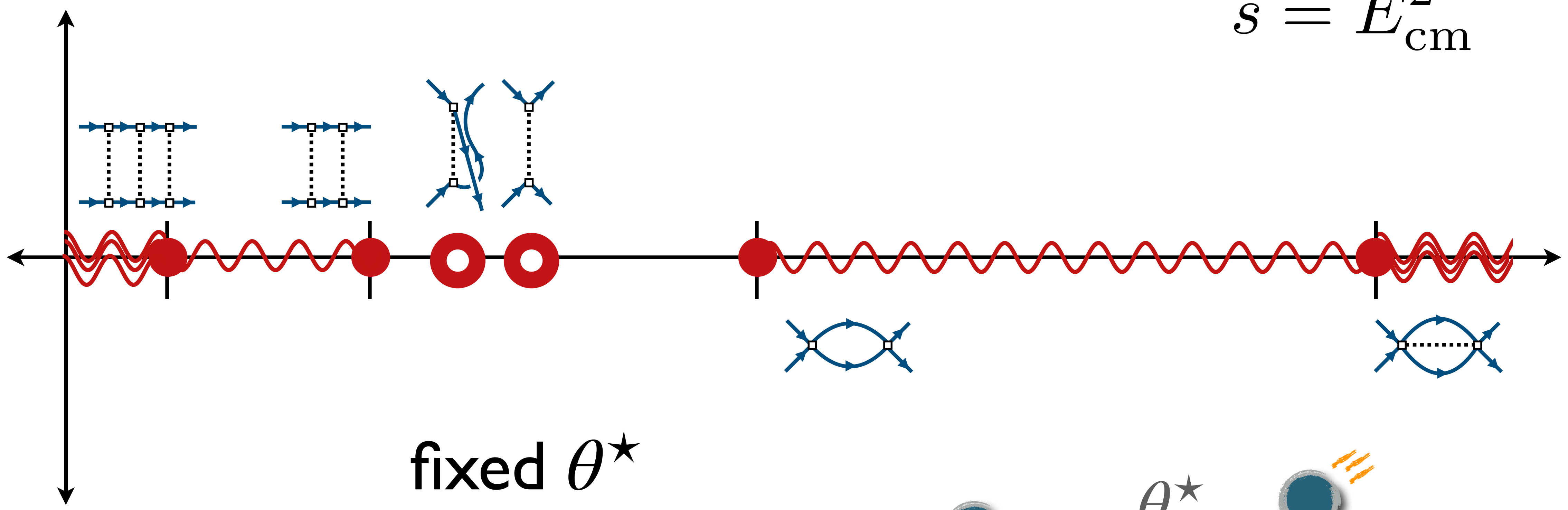
- All orders generic effective theory for nucleons and pions

		masses	
N		M_N	*baryon number conserved*
π		$m_\pi \; (< M_N)$	

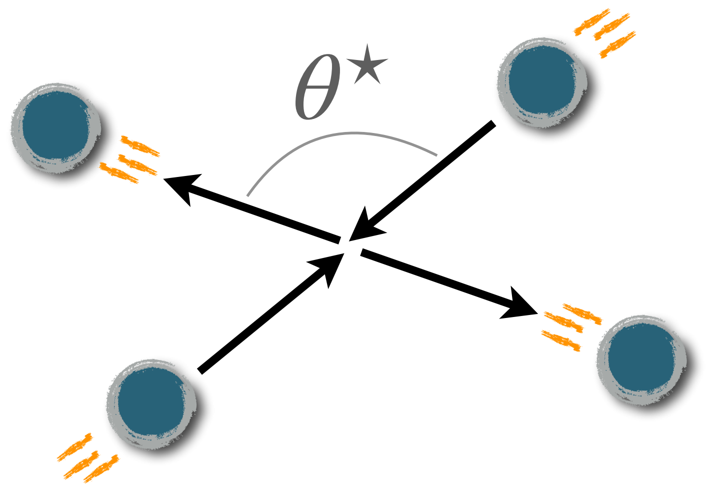


- Analytic structure of the $NN \to NN$ scattering amplitude

$s = E_{\rm cm}^2$



fixed $\theta^\star$



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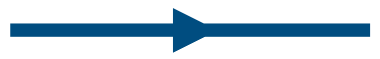
N		M_N	*baryon number conserved*
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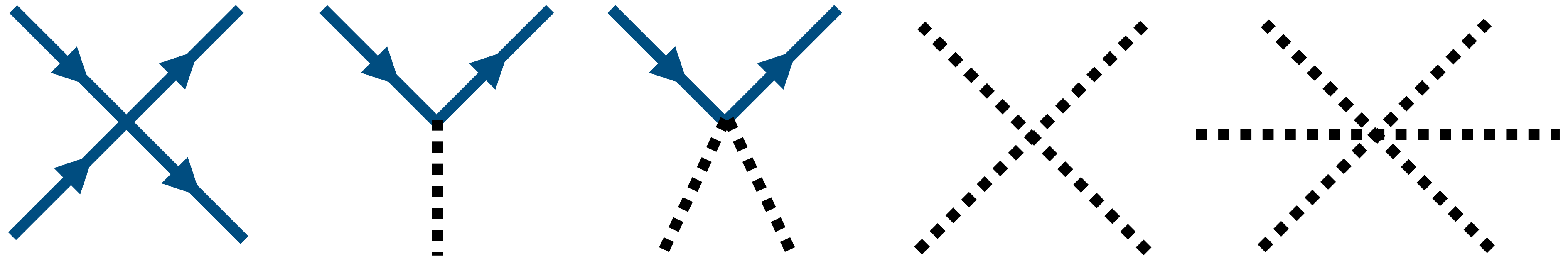

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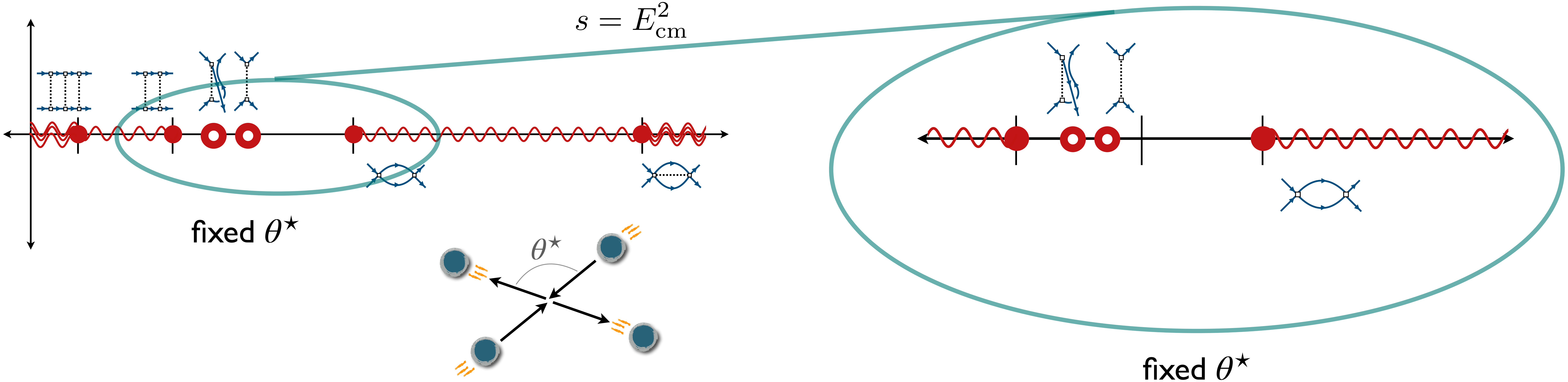

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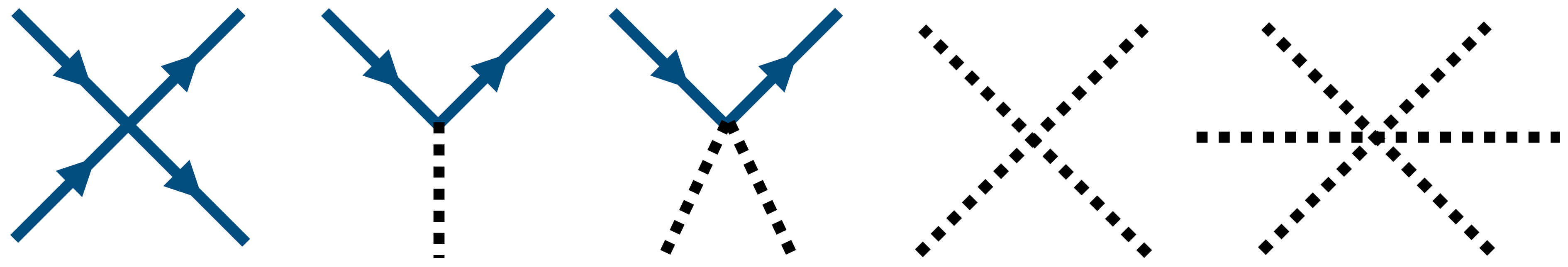

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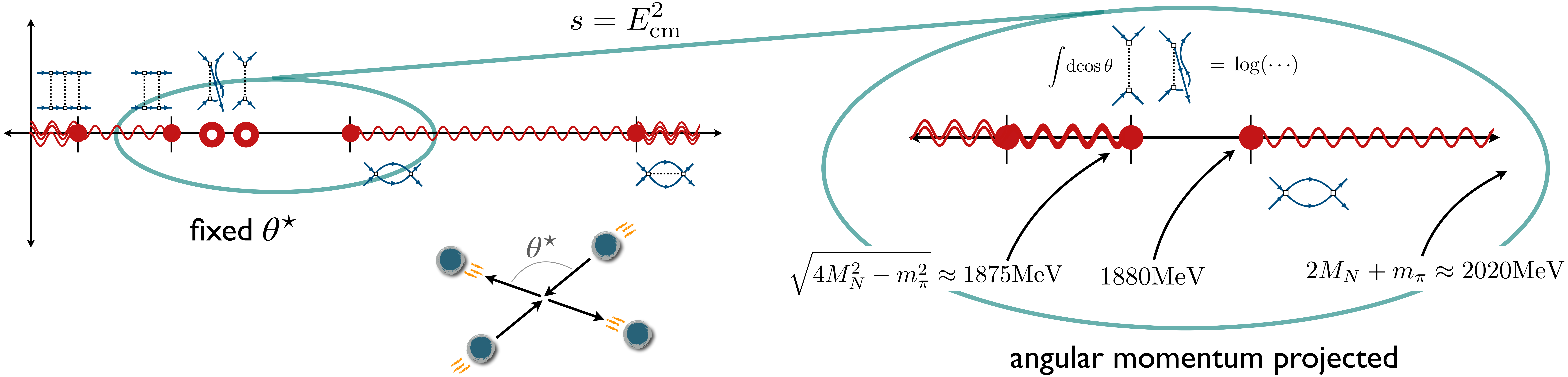
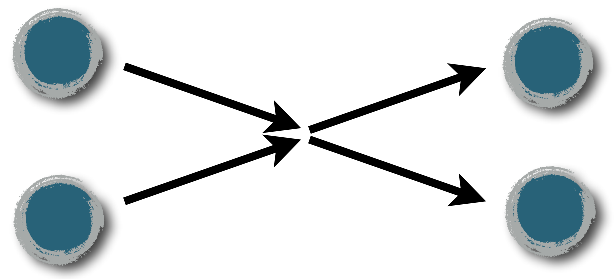


s-channel cut (optical theorem)

- For two-particle scattering energies, how do we know the analytic structure of $\mathcal{M}(s) =$  ?
- The optical theorem tells us…

$$\rho(s)|\mathcal{M}_\ell(s)|^2 = \text{Im}\,\mathcal{M}_\ell(s)$$

where $\rho(s) = \dfrac{\sqrt{1-4M_N^2/s}}{32\pi}$ is the two-particle phase space

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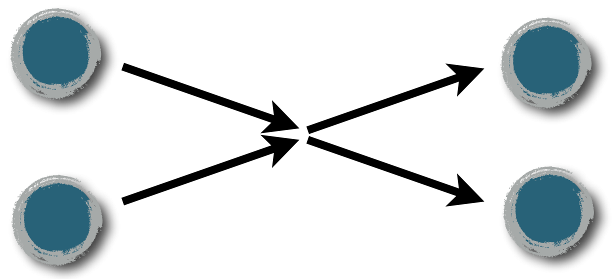
- For two-particle scattering energies, how do we know the analytic structure of $\mathcal{M}(s) =$  ?
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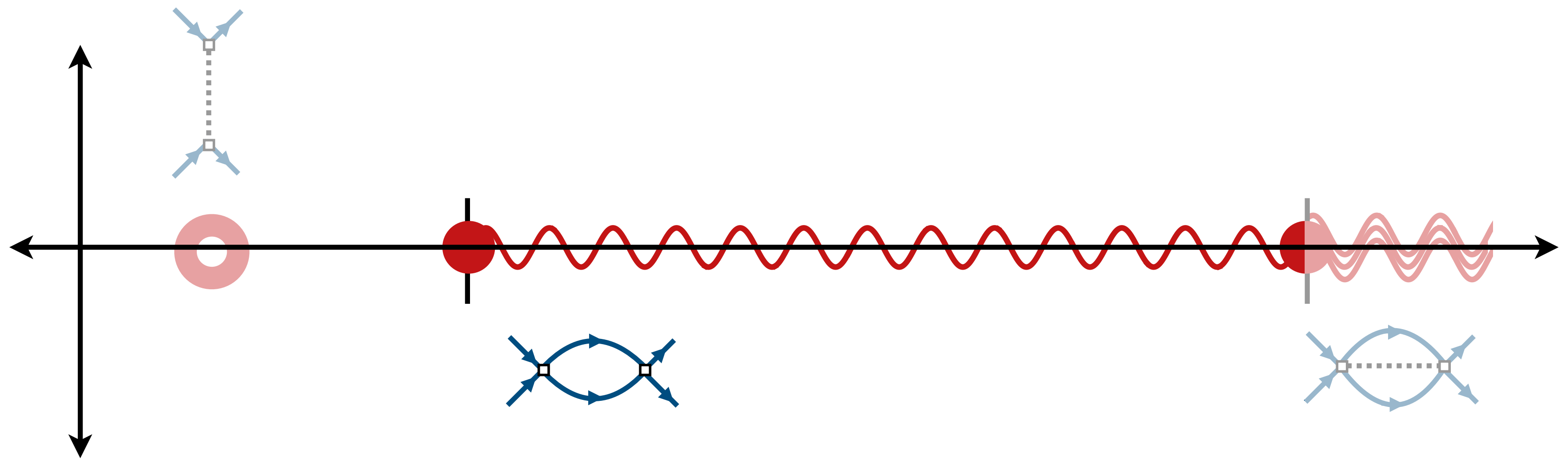


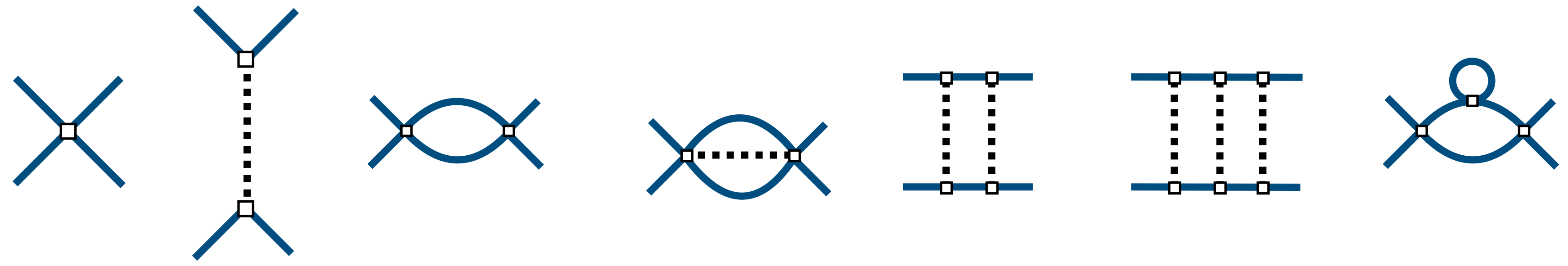
- Unique solution is… $\mathcal{M}_\ell(s) = \dfrac{1}{\mathcal{K}_\ell(s)^{-1} - i\rho(s)}$

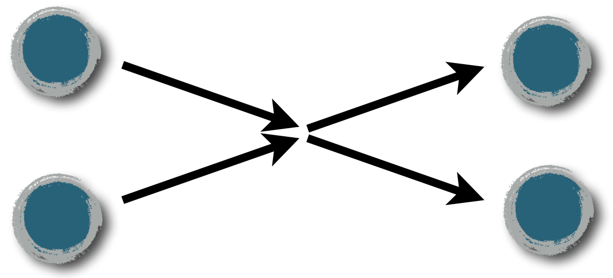
K matrix (short distance)

phase-space cut (long distance)

Key message: *s-channel square-root branch cut from the optical theorem*

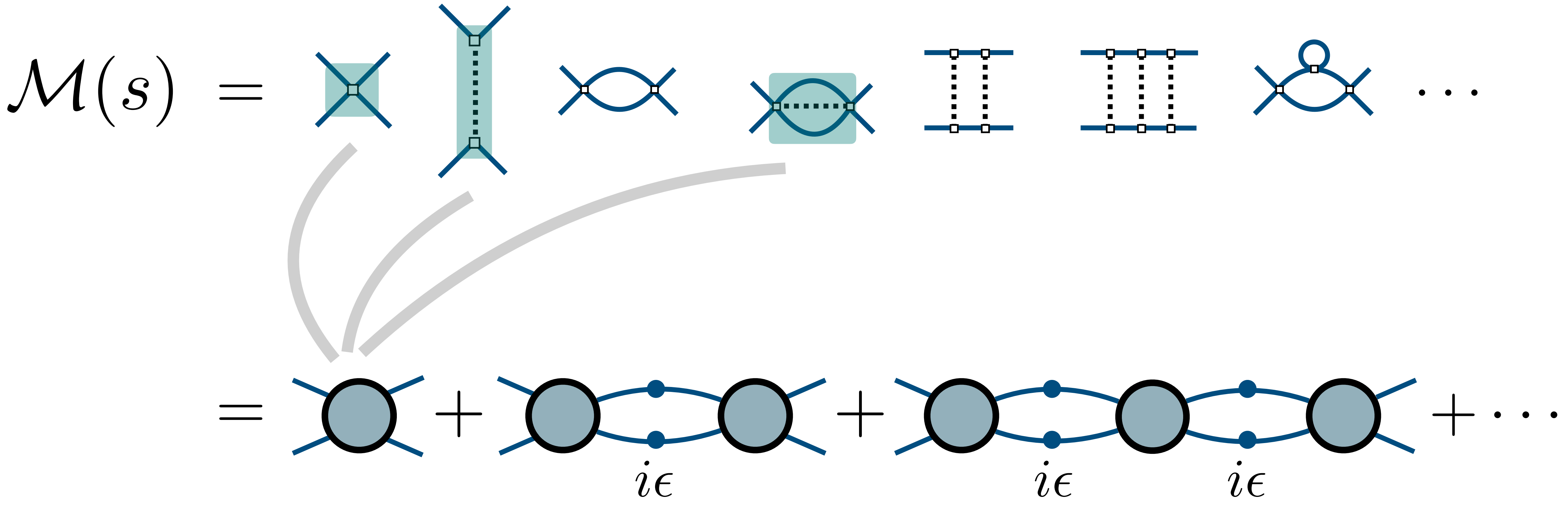
s-channel cut (diagrammatic)

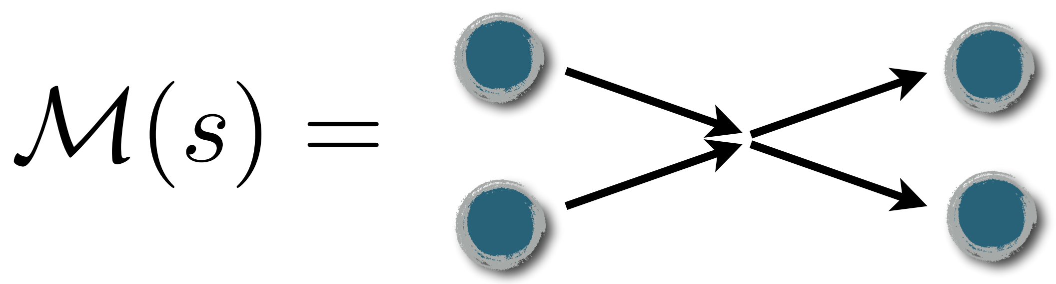
$$\mathcal{M}(s) =$$  $\cdots$

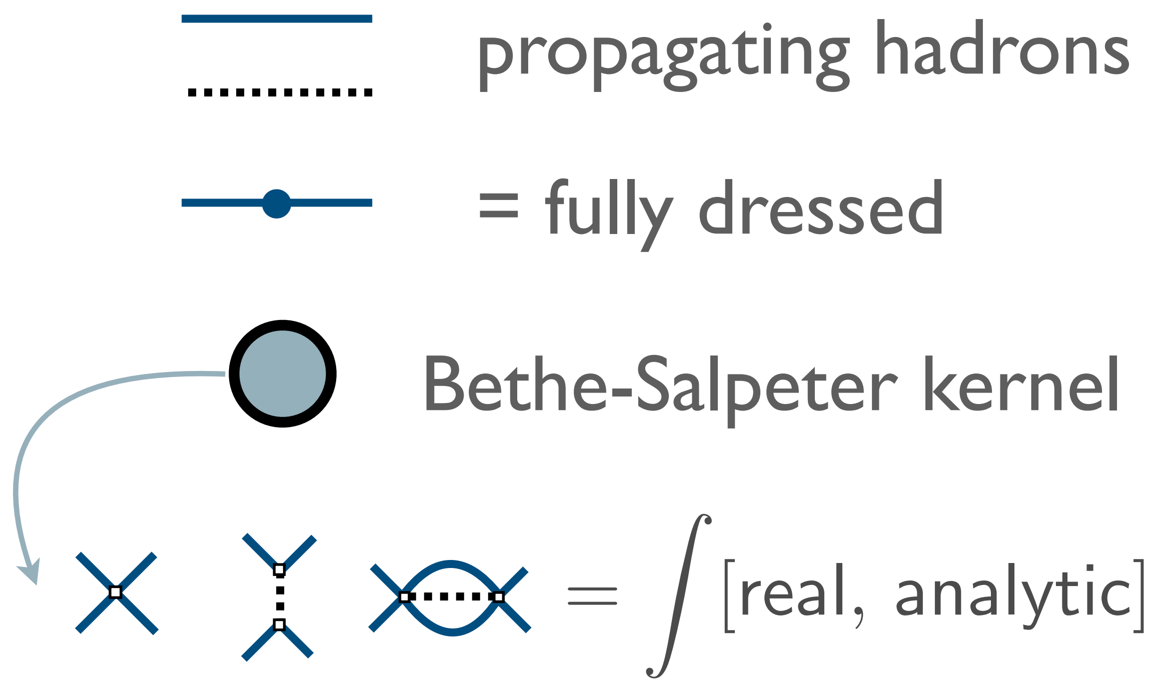
$$\mathcal{M}(s) =$$ 

propagating hadrons

s-channel cut (diagrammatic)



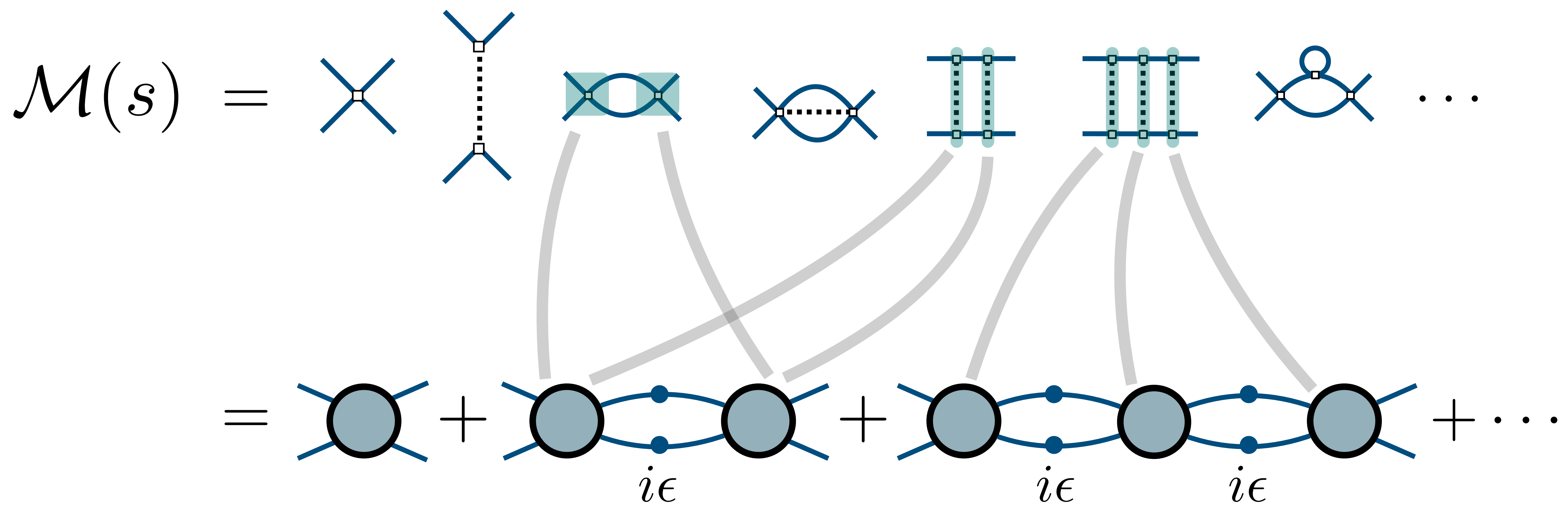


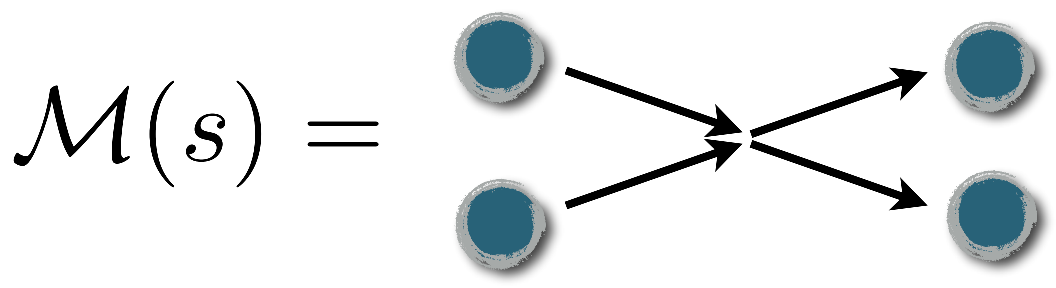


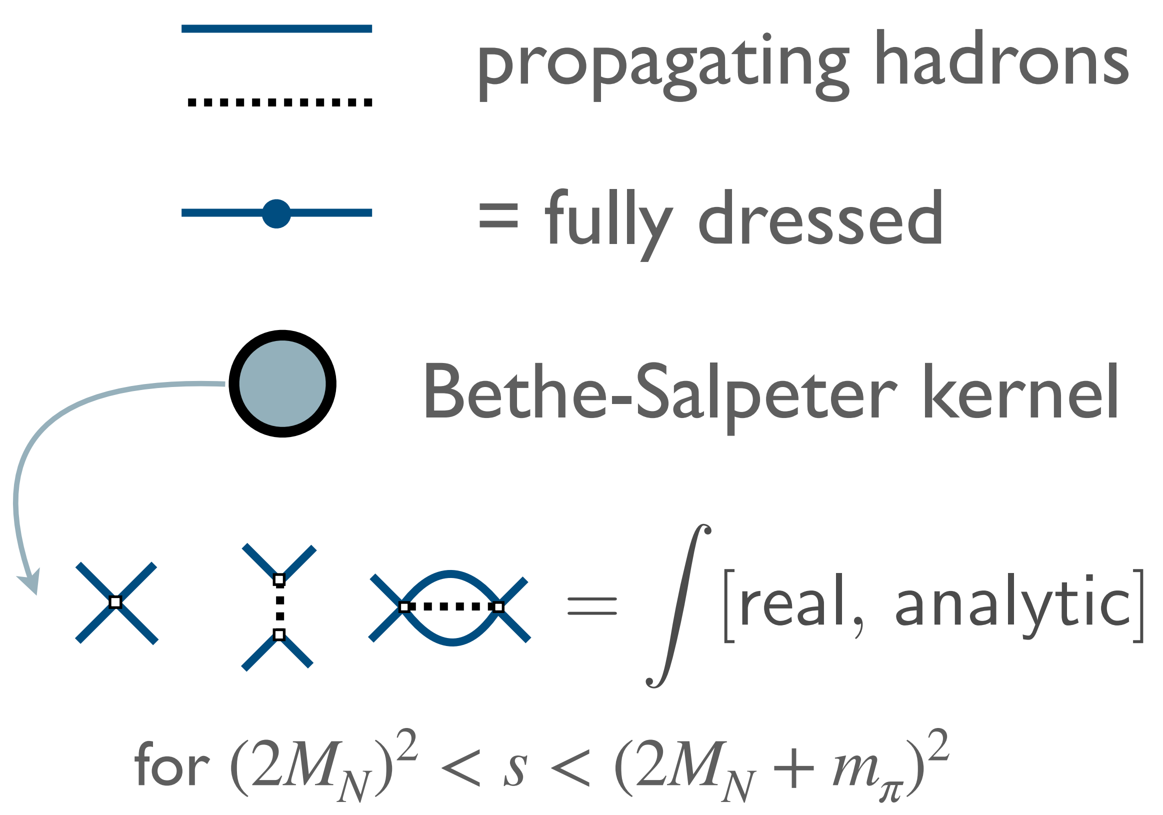
for $(2M_N)^2 < s < (2M_N + m_\pi)^2$

Framework for generic, EFT independent, all-orders diagrammatic relations

s-channel cut (diagrammatic)







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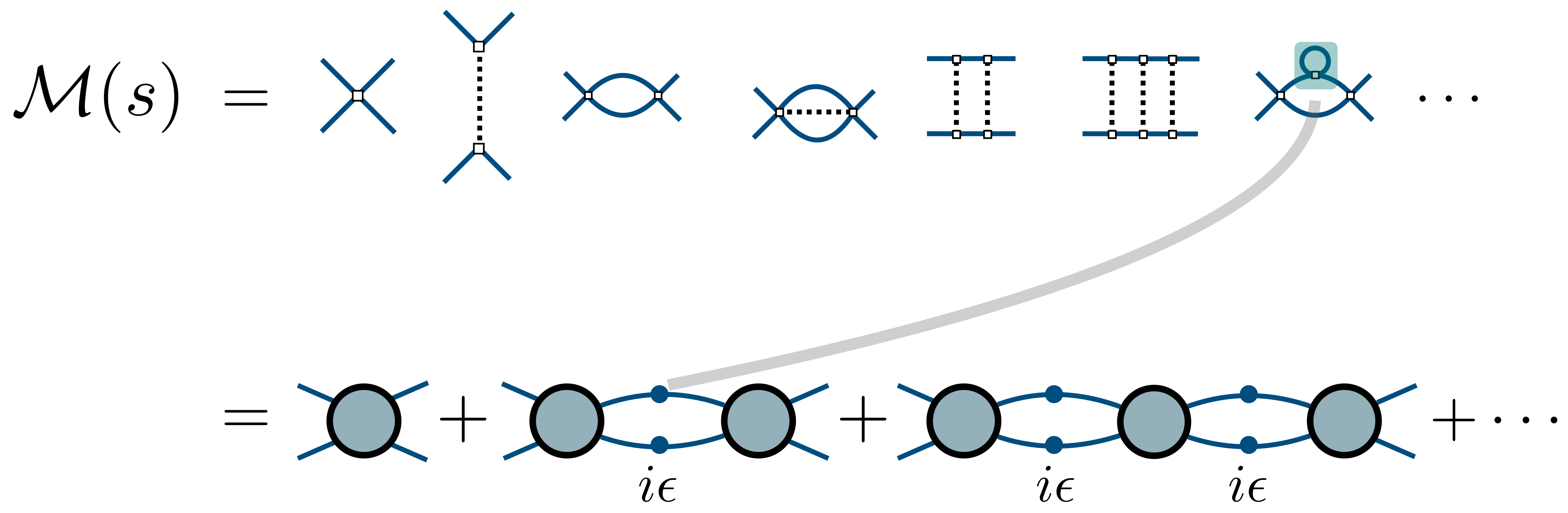

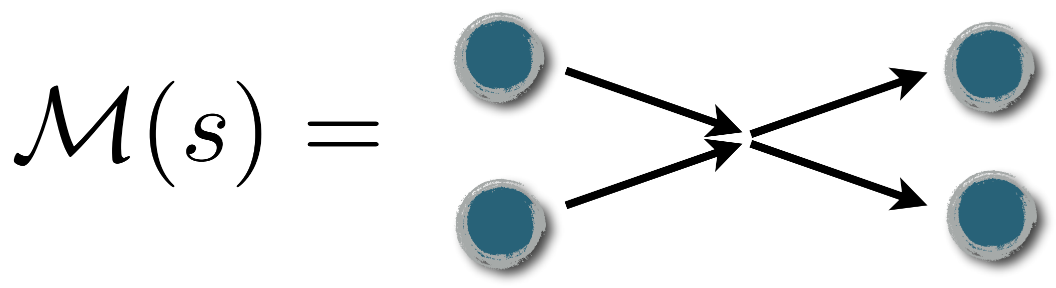

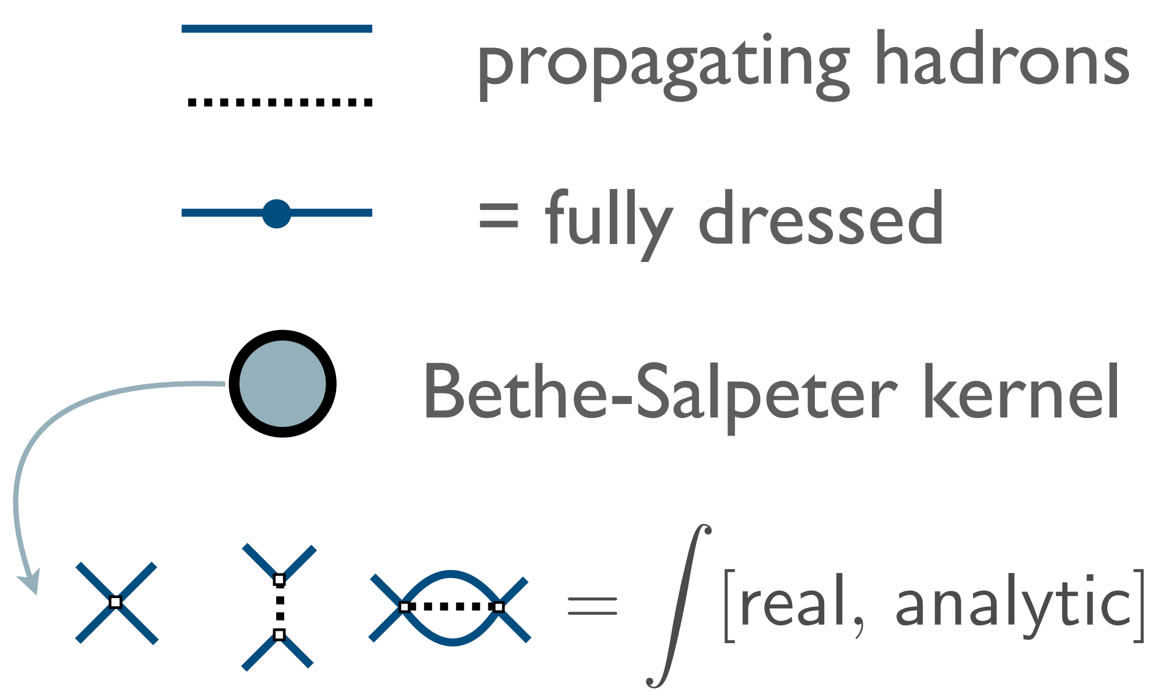

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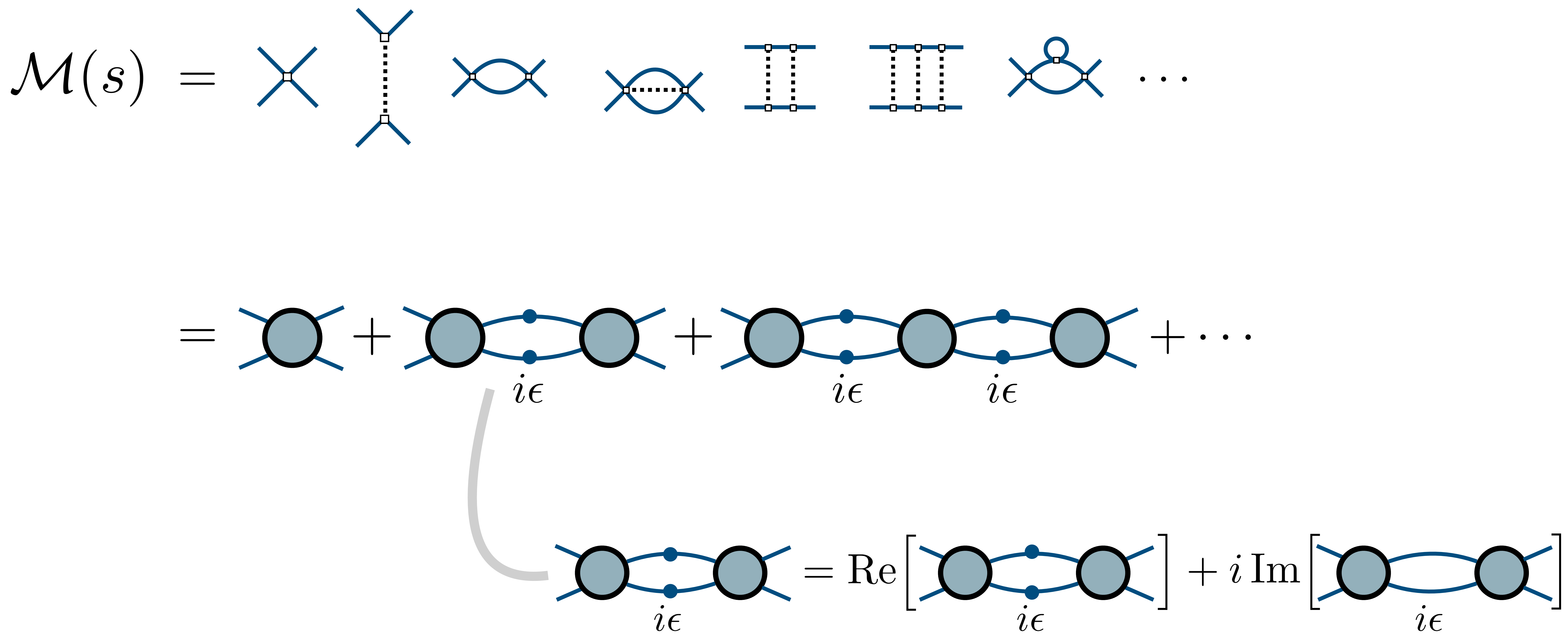

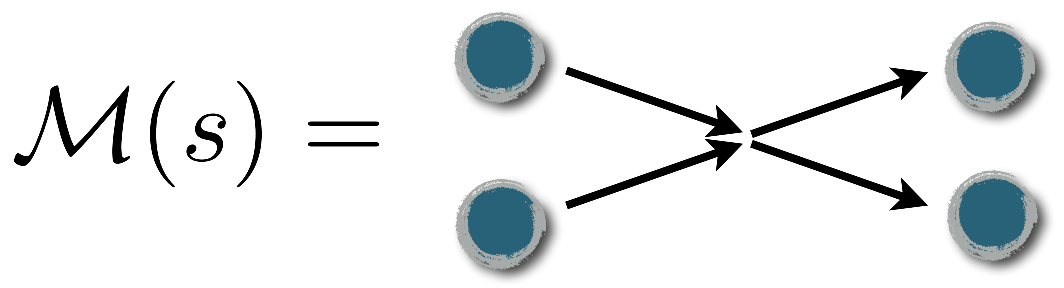

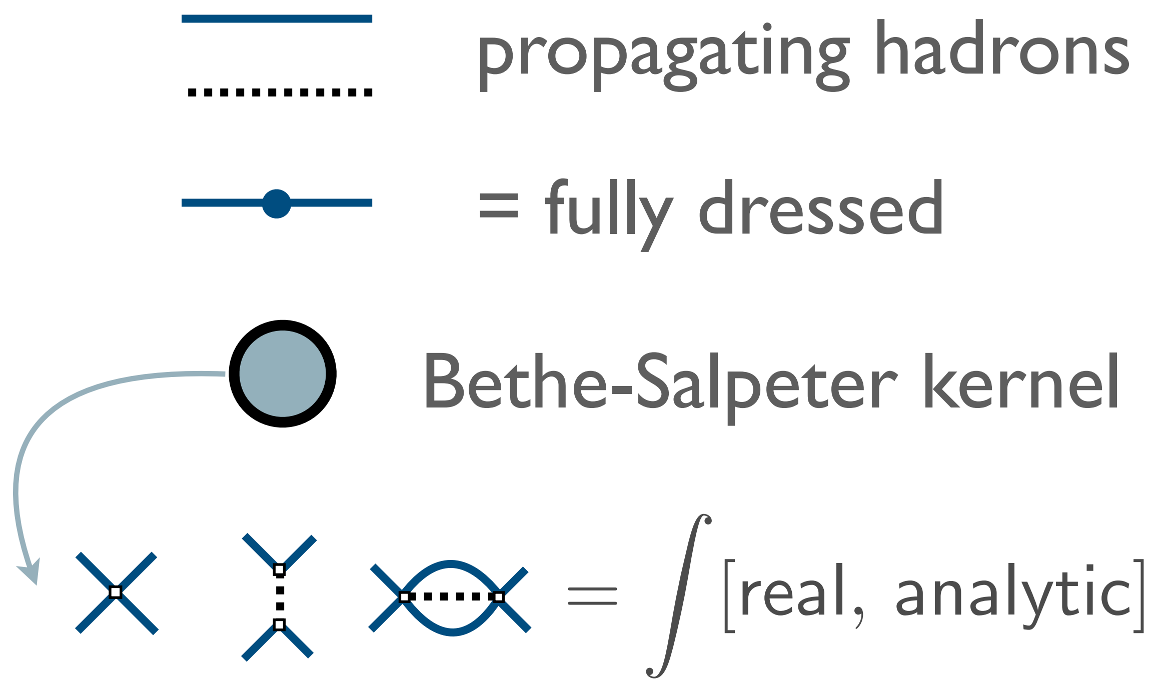

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Framework for generic, EFT independent, all-orders diagrammatic relations

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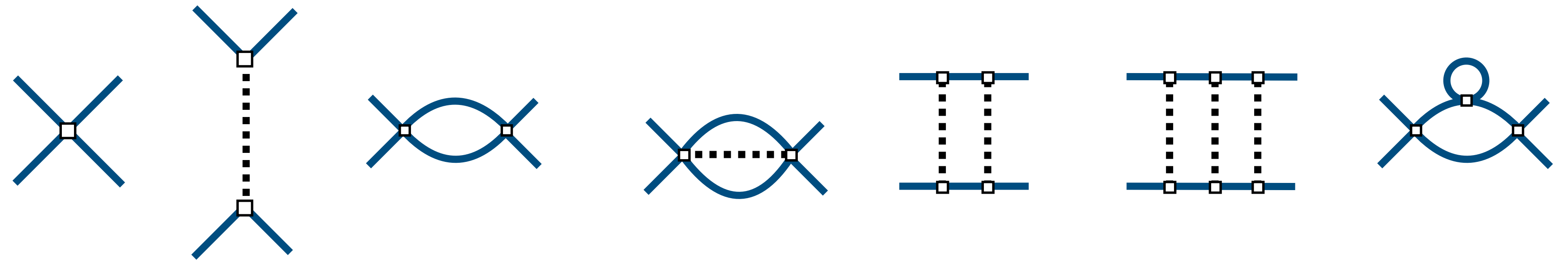
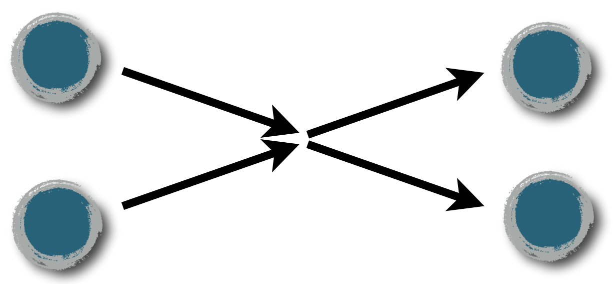
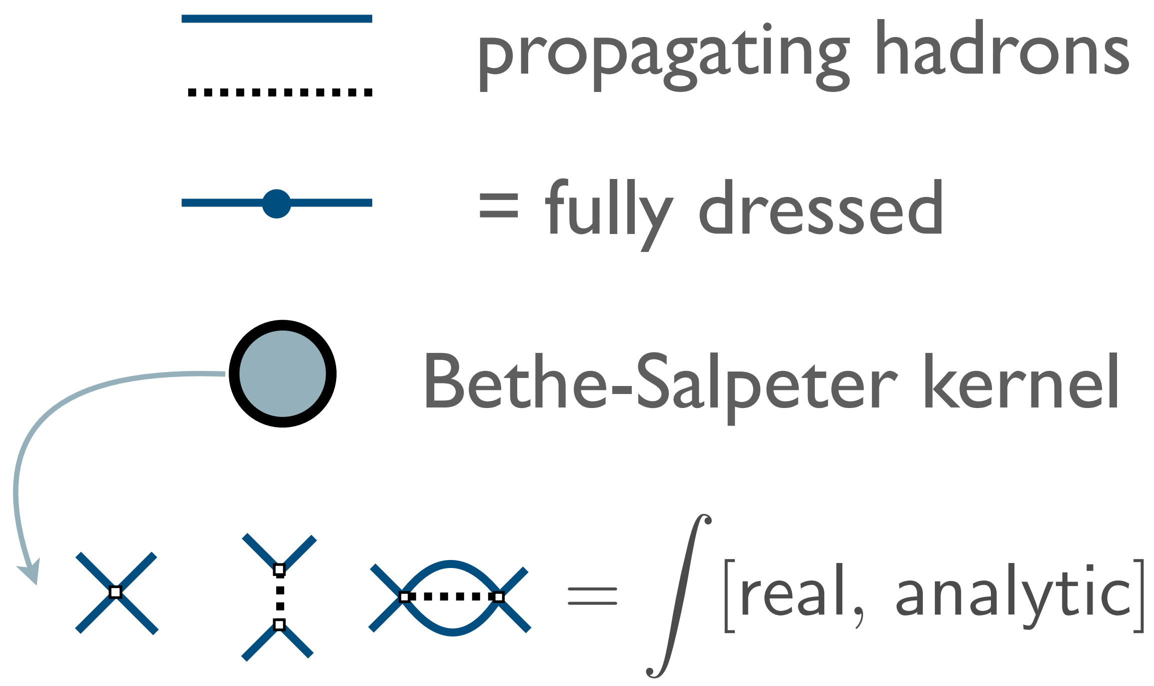
$\mathcal{M}(s) =$  $\cdots$

$\mathcal{M}(s) =$ 



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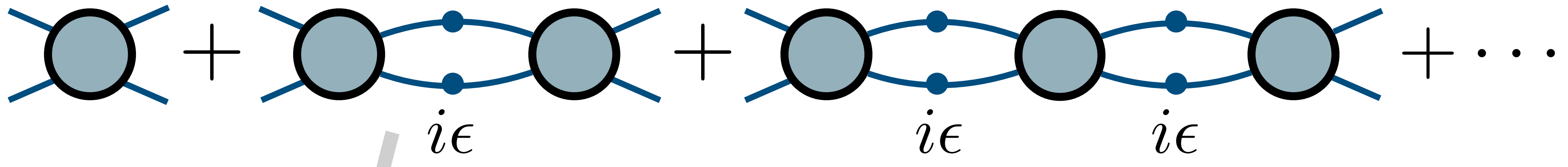
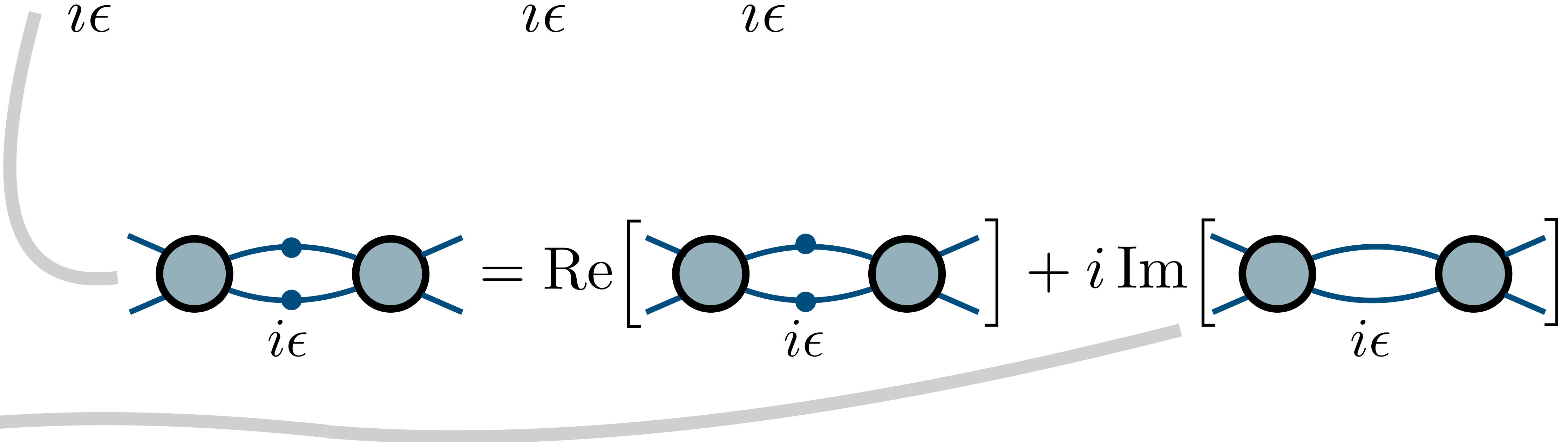
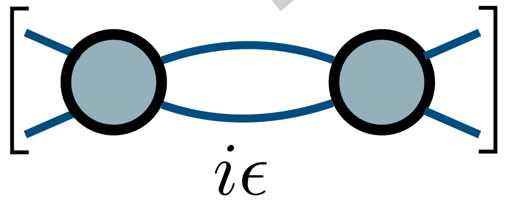
$=$ 

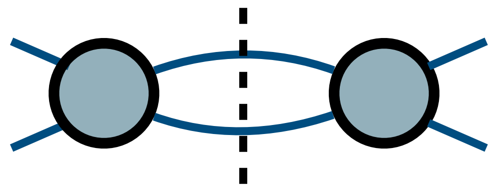


Framework for generic, EFT independent, all-orders diagrammatic relations

$i\,\mathrm{Im}$  $= i\pi \int_{\boldsymbol{k}} \frac{\delta((P-k)^2 - m^2)}{2\omega_{\boldsymbol{k}}} B_{\ell m}(|\boldsymbol{k}_{\mathrm{cm}}|)^2$

$$\boldsymbol{k}^2_{\mathrm{cm}} \stackrel{!}{=} s/4 - M_N^2 \equiv p(s)^2$$

imaginary part is set exactly on-shell

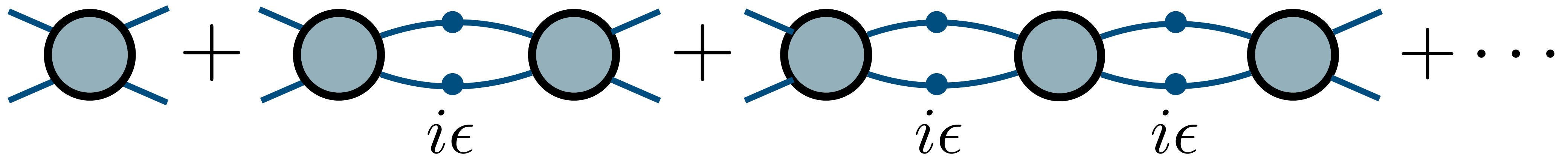
$= B_{\ell m}(s)\; i\rho(s)\; B_{\ell m}(s) =$ 

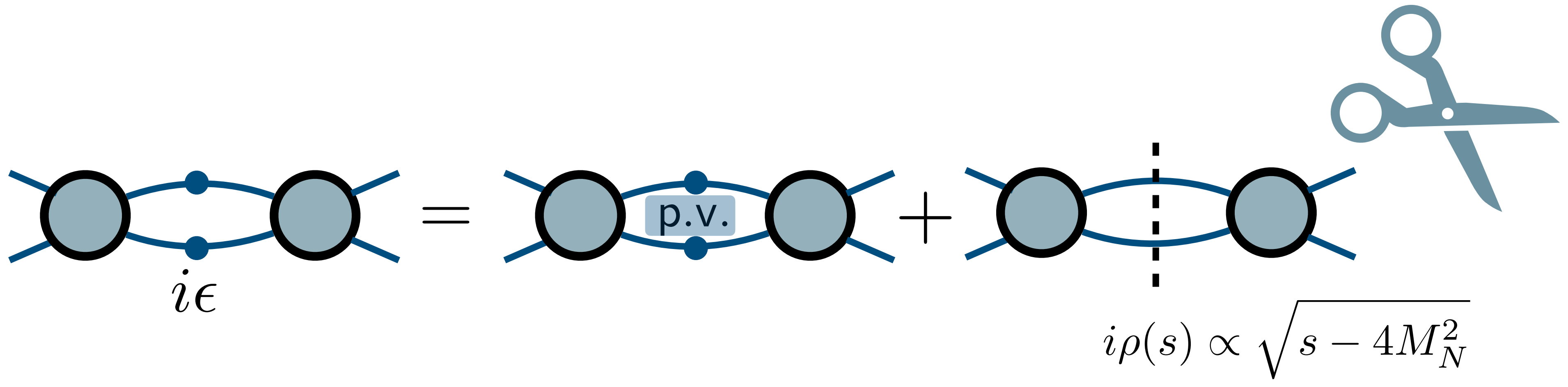
$i\rho(s) \propto \sqrt{s - 4M_N^2}$

$$\rho(s) = \frac{\sqrt{1 - 4M_N^2/s}}{32\pi}$$

s-channel cut (diagrammatic)

$\mathcal{M}(s) =$





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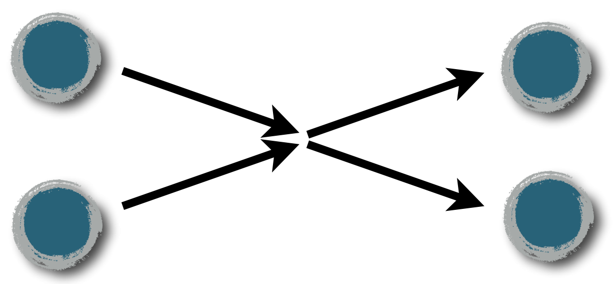
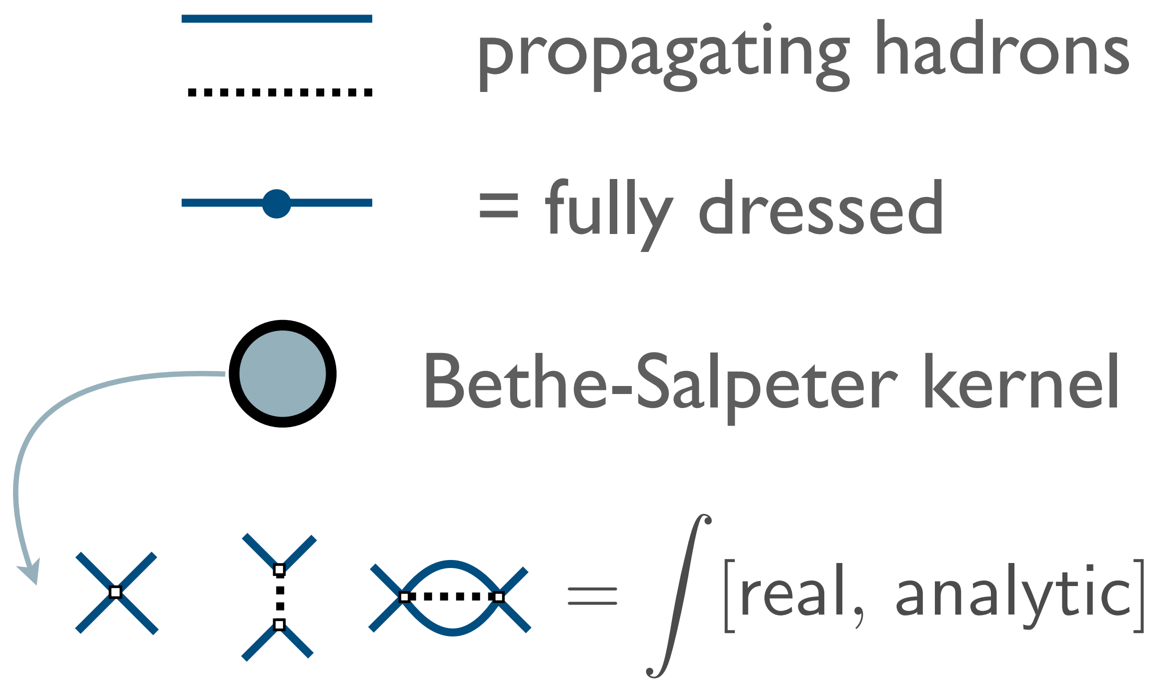




for $(2M_N)^2 < s < (2M_N + m_\pi)^2$

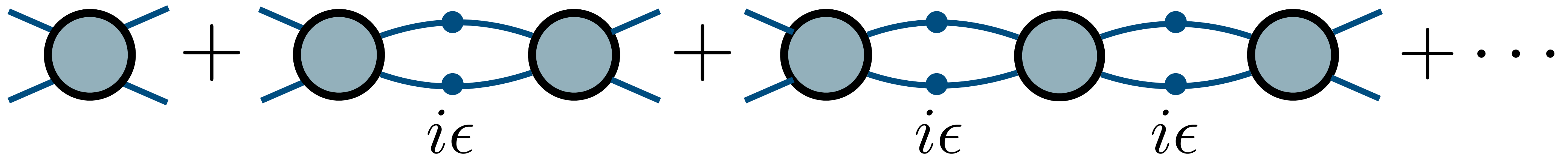
Framework for generic, EFT independent, all-orders diagrammatic relations

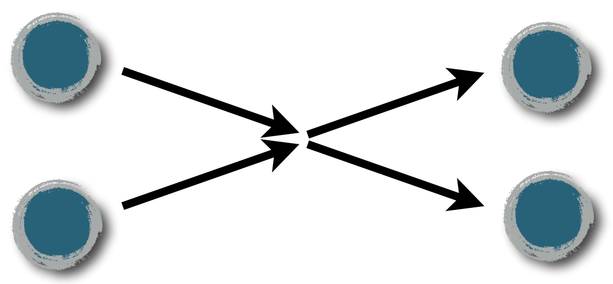
$$\boldsymbol{k}_{\text{cm}}^2 \overset{!}{=} s/4 - M_N^2 \equiv p(s)^2$$

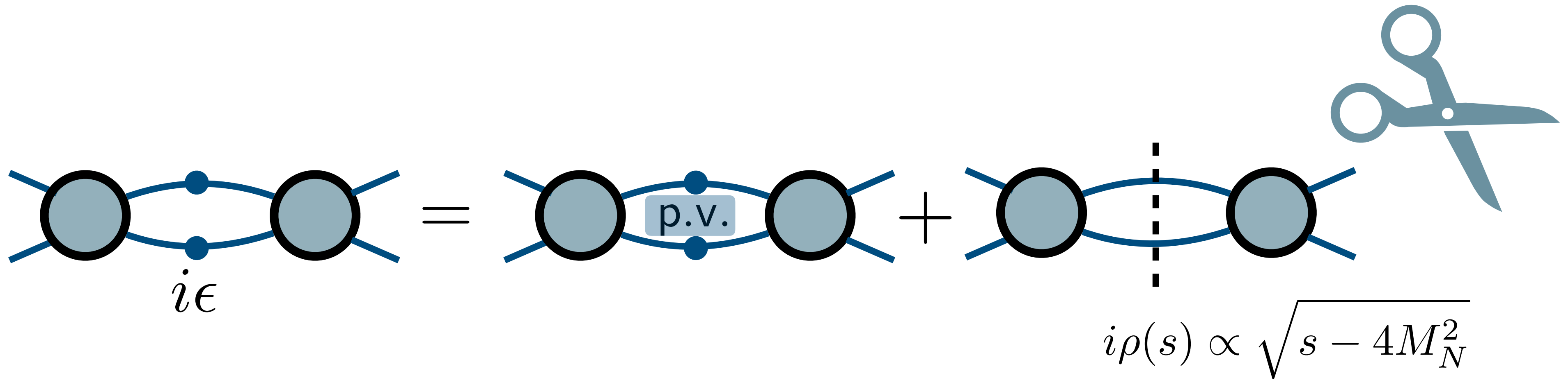
imaginary part is set exactly on-shell

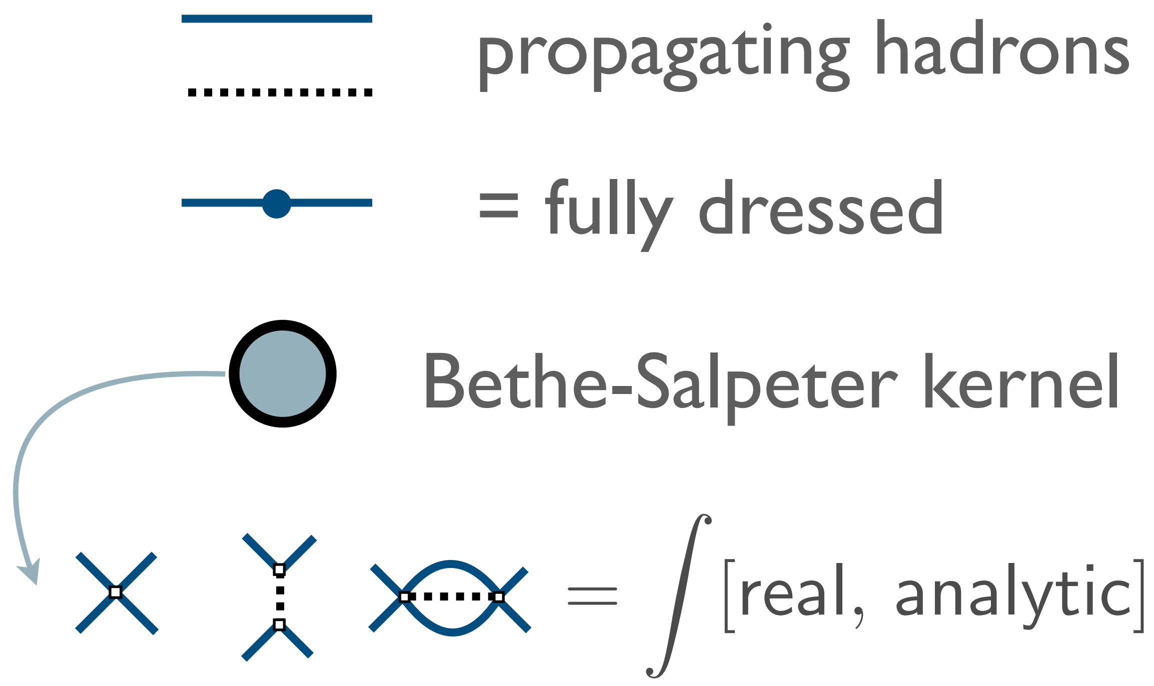
$$\rho(s) = \frac{\sqrt{1 - 4M_N^2/s}}{32\pi}$$

s-channel cut (diagrammatic)

$\mathcal{M}(s) =$ 

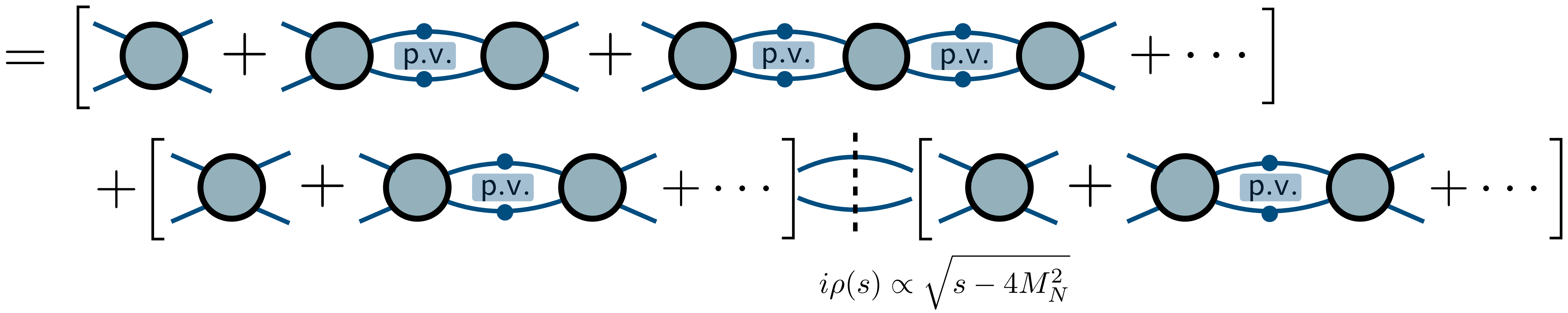
$\mathcal{M}(s) =$ 





for $(2M_N)^2 < s < (2M_N + m_\pi)^2$

defines the K matrix $\mathcal{K}(s)$



$$= \mathcal{K}(s) + \mathcal{K}(s) i\rho(s) \mathcal{K}(s) + \cdots = \frac{1}{\mathcal{K}(s)^{-1} - i\rho(s)}$$

Framework for generic, EFT independent, all-orders diagrammatic relations

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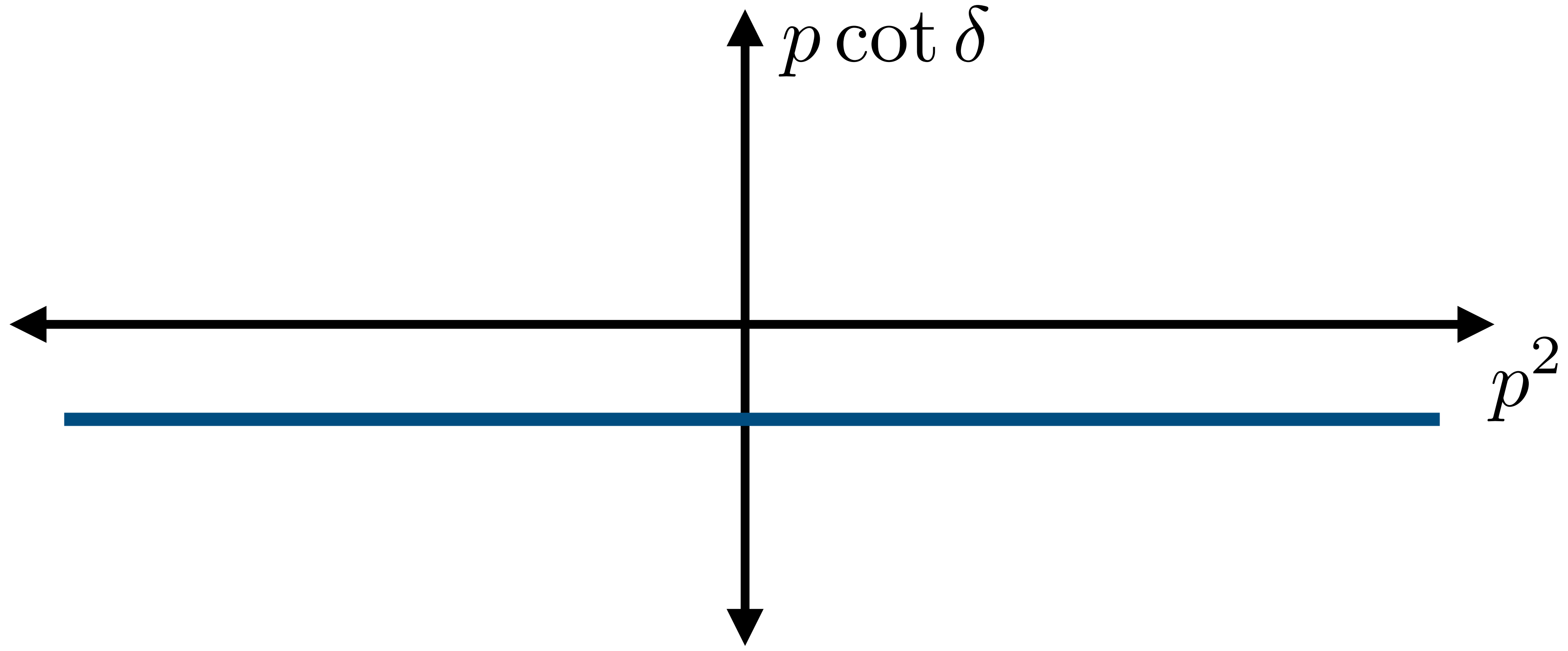
Continuation below threshold

$$\rho(s) = \frac{\sqrt{1-4M_N^2/s}}{32\pi} \qquad p = \sqrt{s/4 - M_N^2}$$

$$\mathcal{M}(s) = \frac{1}{\mathcal{K}(s)^{-1} - i\rho(s)} = \frac{16\pi\sqrt{s}}{p\cot\delta(p) - ip}$$

- $p\cot\delta(p)$ has a smooth continuation below threshold   example: $p\cot\delta(p) = C_0 = -1/a$
- For ip we must choose a Riemann sheet



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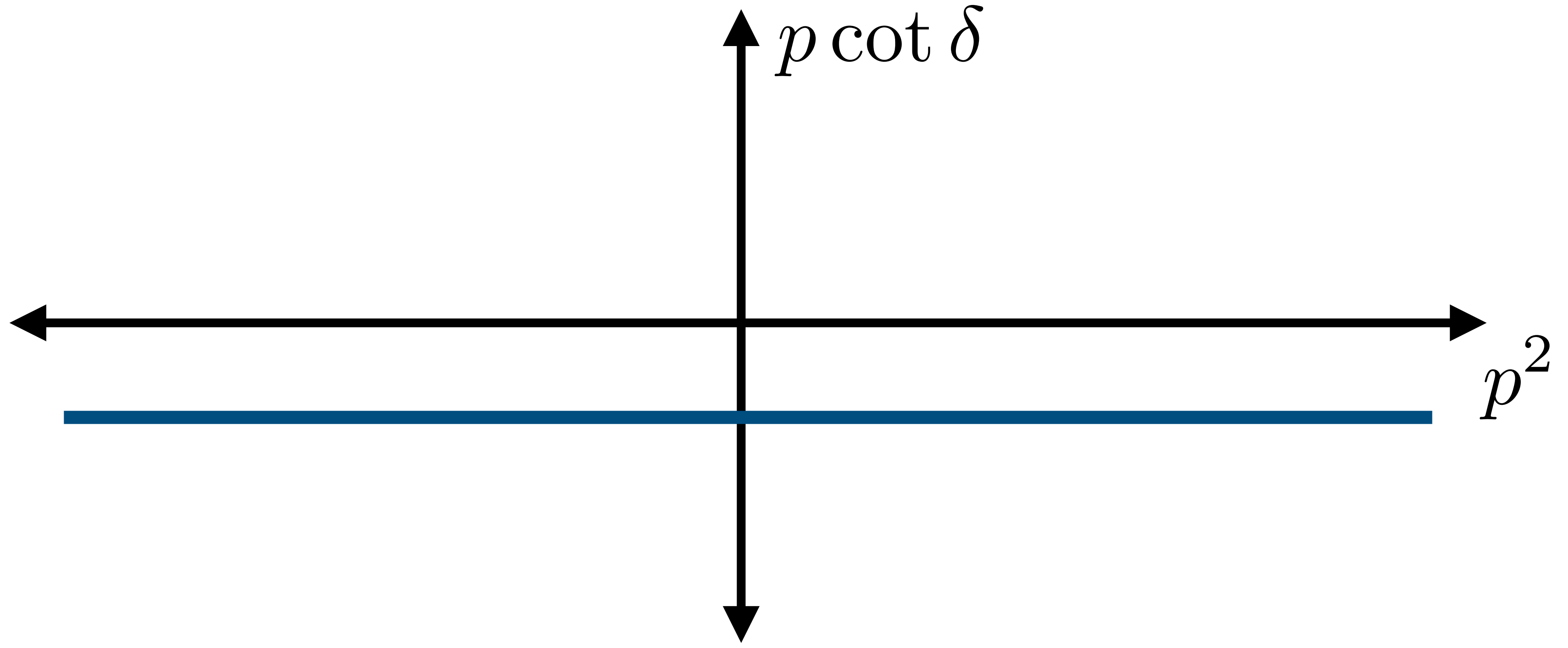
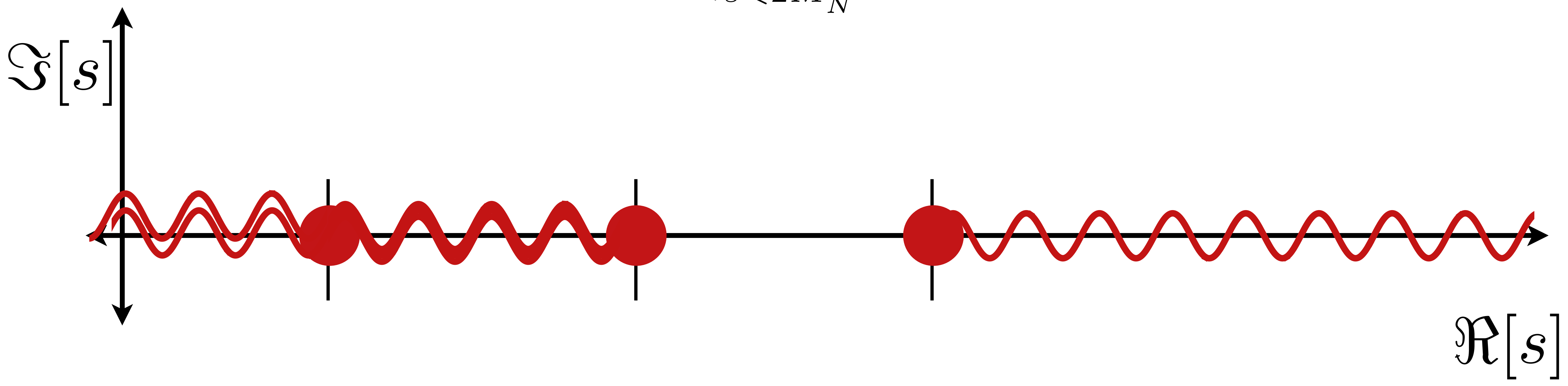


"Physical sheet" / First sheet

$$\mathrm{Im}[p] > 0 \quad \leftrightarrow \quad -ip\Big|_{s<2M_N^2} = |p| = \sqrt{-p^2}$$



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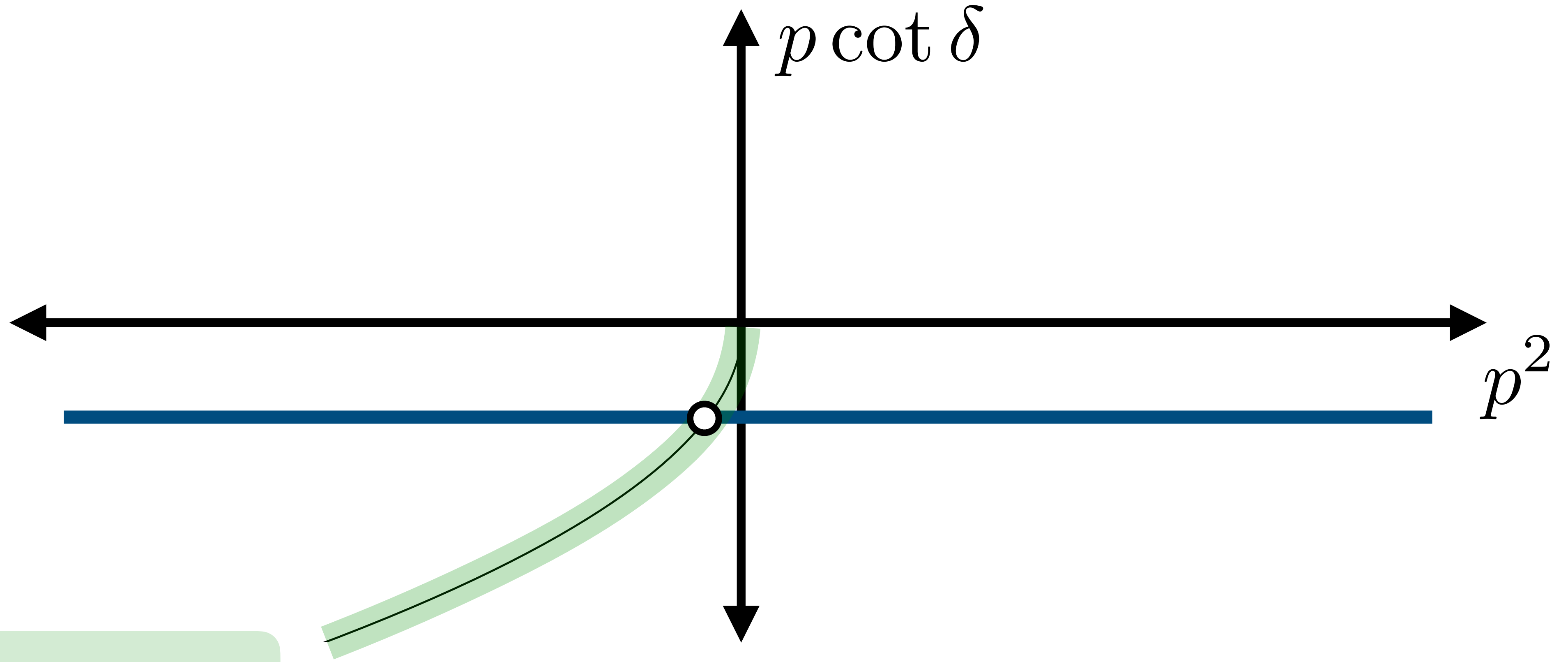

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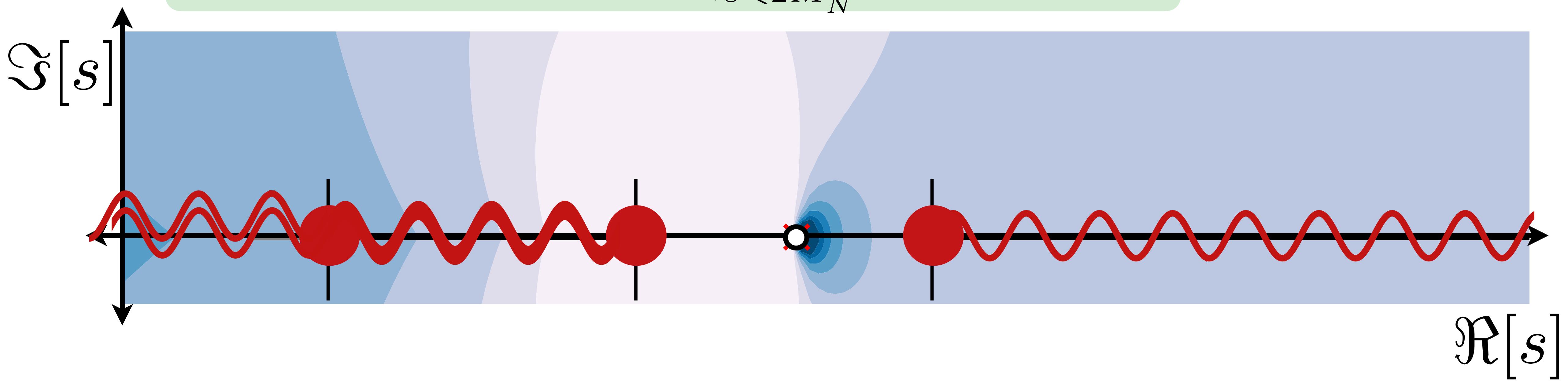

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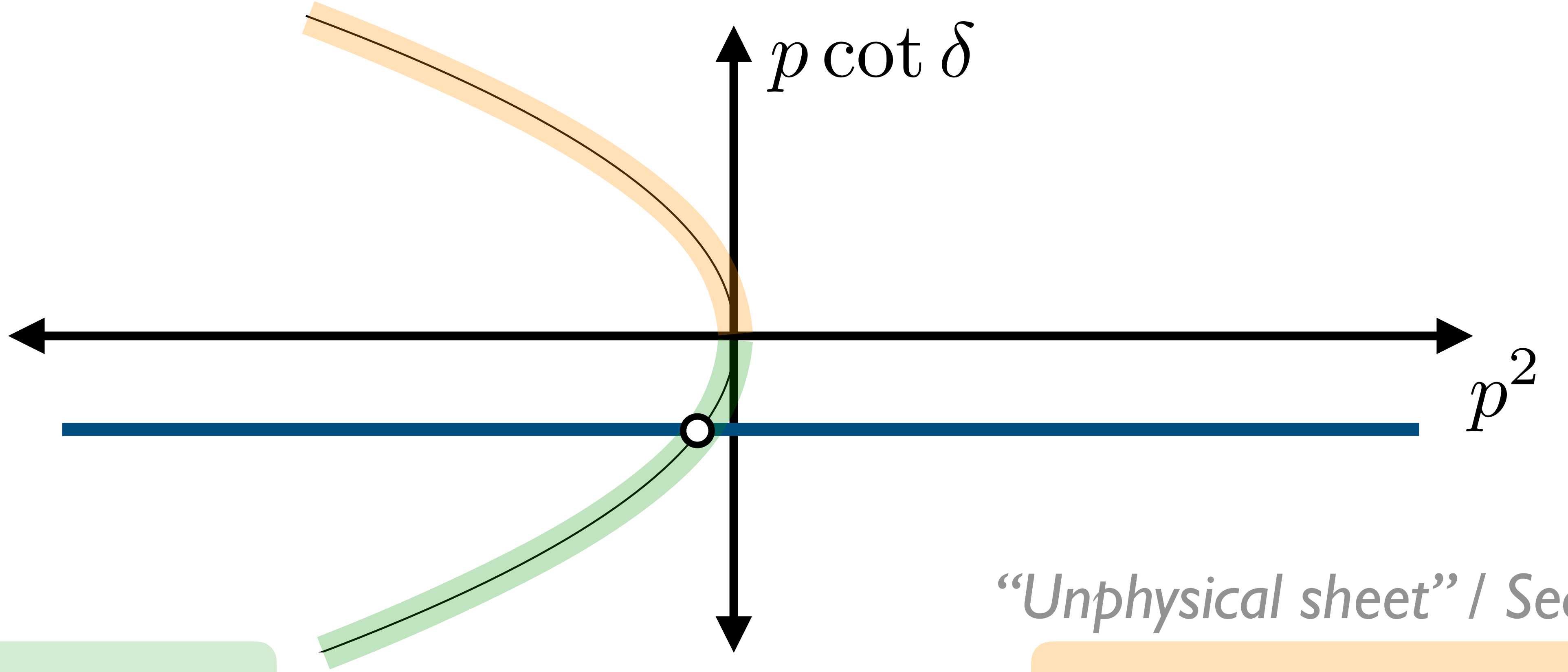

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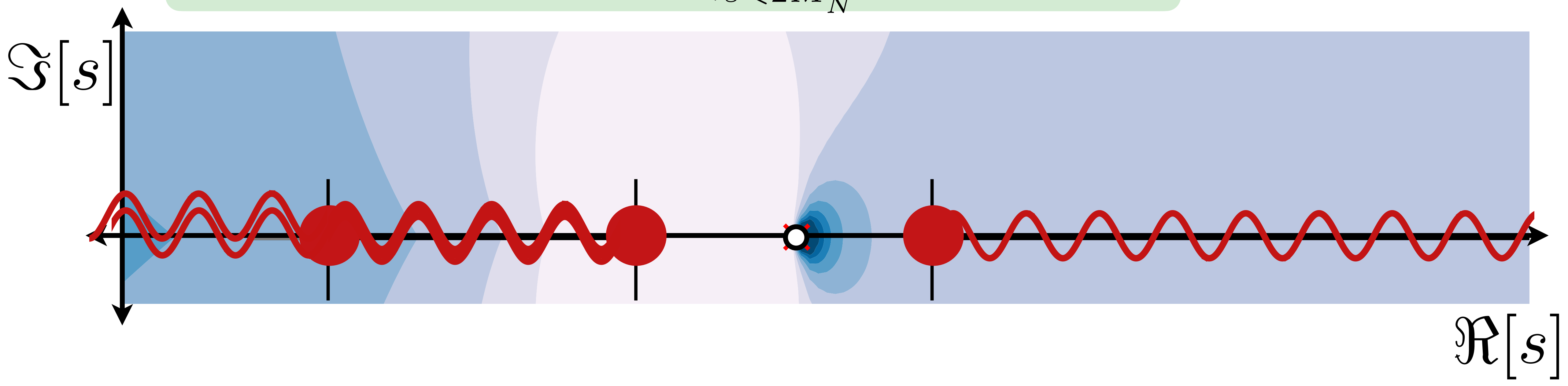
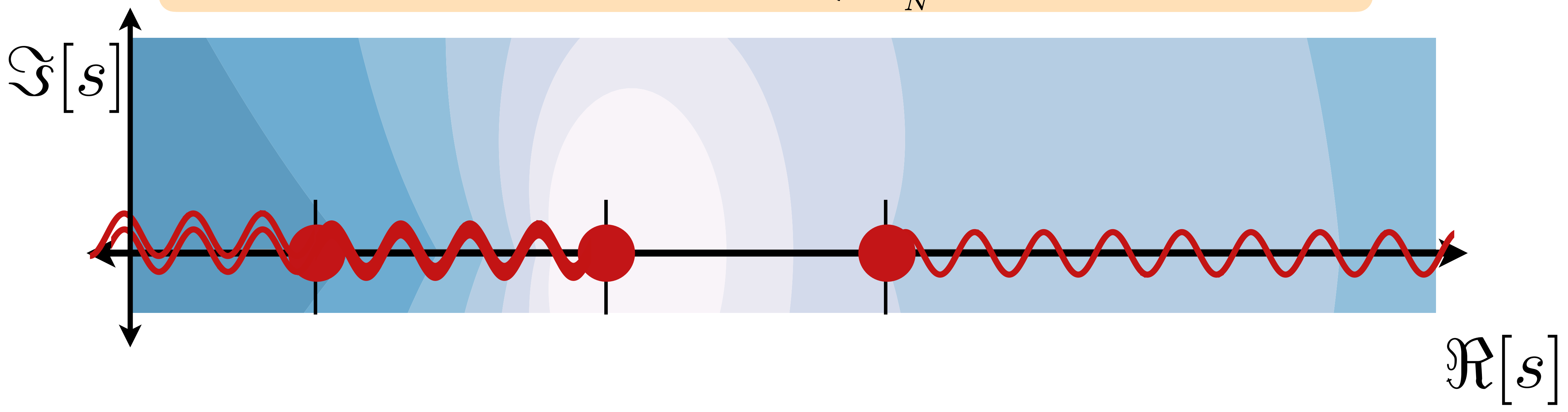


"Unphysical sheet" / Second sheet

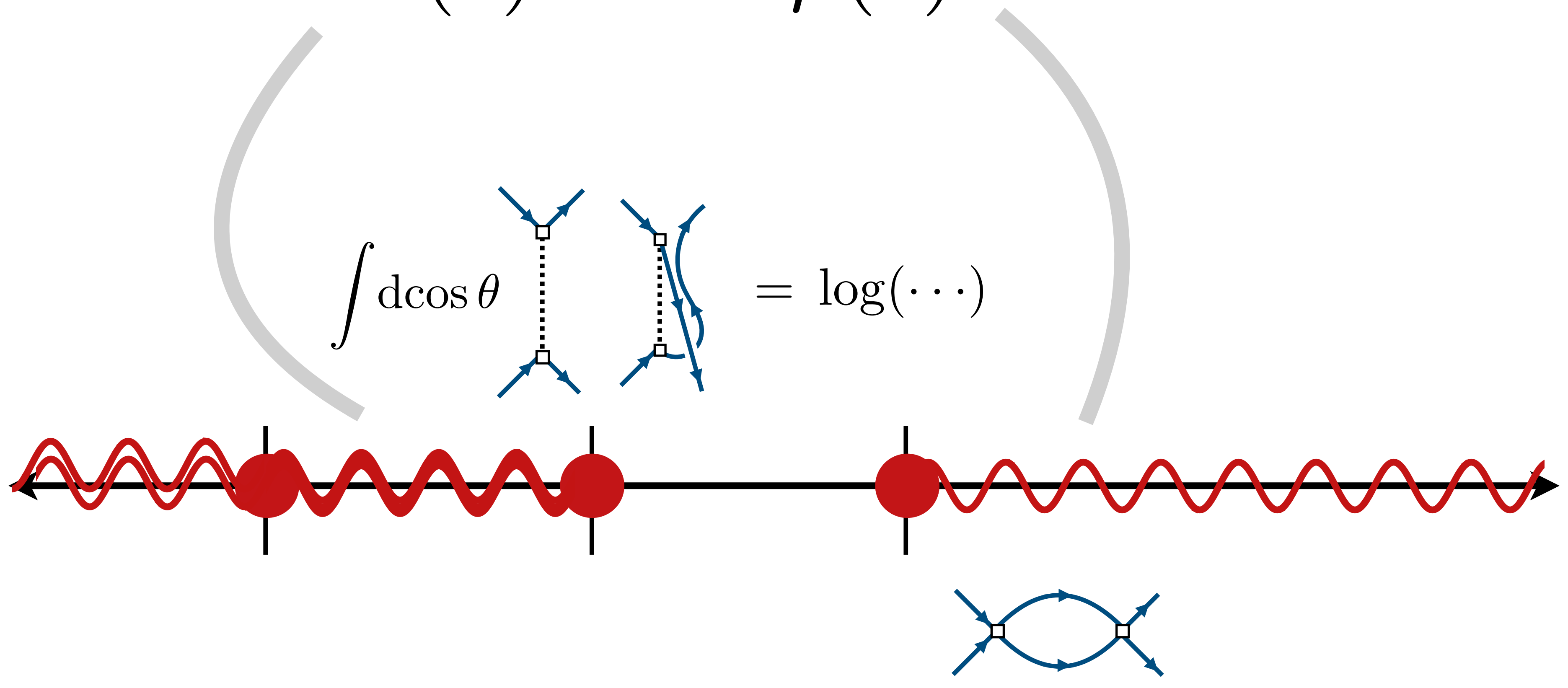
$$\mathrm{Im}[p] < 0 \quad \longleftrightarrow \quad -ip\Big|_{s<2M_N^2} = -|p| = -\sqrt{-p^2}$$



Infinite-volume review

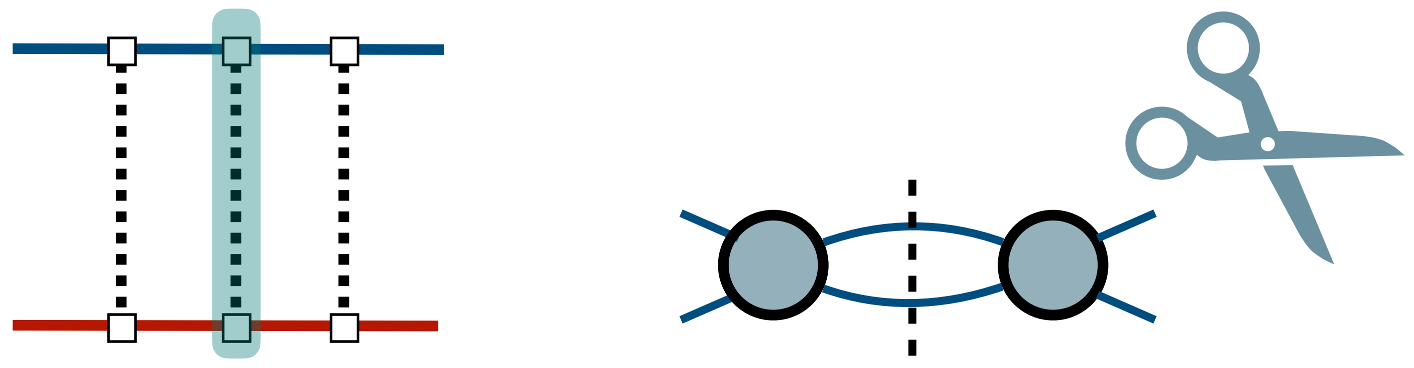
- s-channel cut arises from the imaginary part of two-particle loops
  - related to continuum of two-particle scattering states
  - approach the pole $\rightarrow$ need $i\epsilon$
  - Dirac delta enforces physical energy (on-shell)

$$\mathcal{M}(s) = \frac{1}{\mathcal{K}(s)^{-1} - i\rho(s)}$$



- Sub-threshold continuation requires
  - Riemann sheet specification: $i\rho(s) \rightarrow \pm\,|\rho(s)|$
  - near-threshold analyticity of $K(s)$

- Recipe to analytically continue:
  - break into on-shell s-channnel cuts
  - continue each contribution (choosing a sheet for $\rho(s)$)
  - all exchanges contribute to the nearest left-hand cut

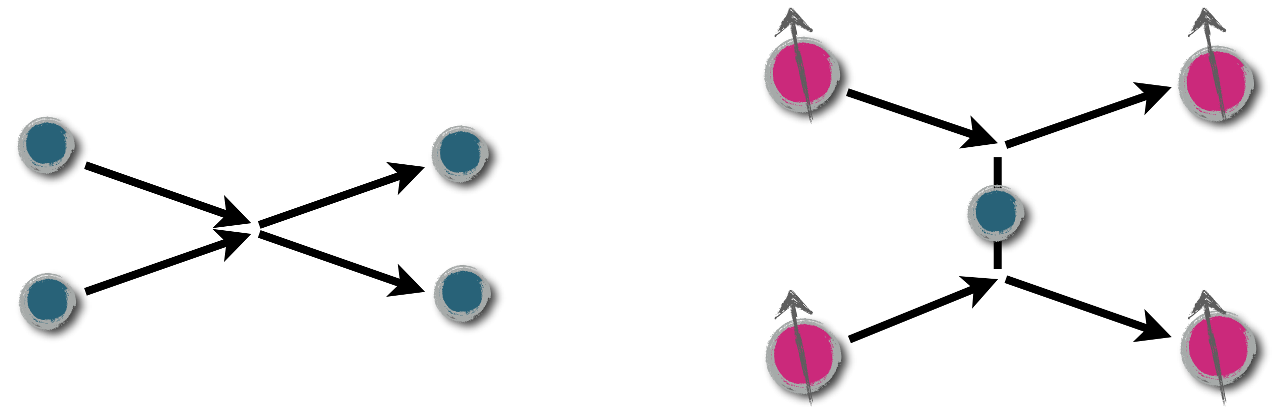


…switching now to the finite-volume…

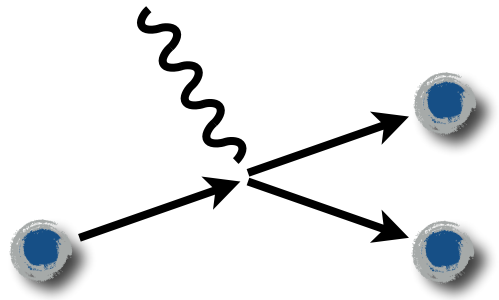
Landscape of amplitudes

☐ Large body of work dedicated to extracting amplitudes from finite-volume data

- ☑ Two-to-two scattering: $2 \to 2$



- ☑ Decays with an external current: $1 \xrightarrow[\mathcal{J}]{} 2$



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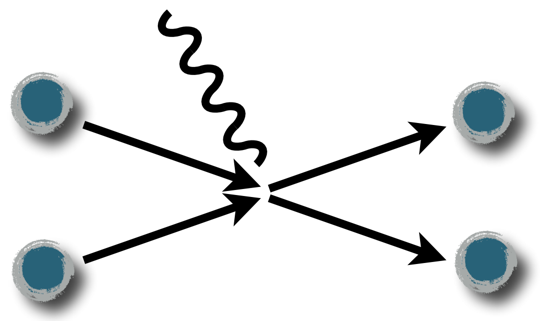
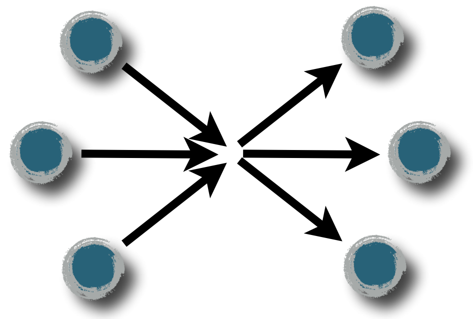
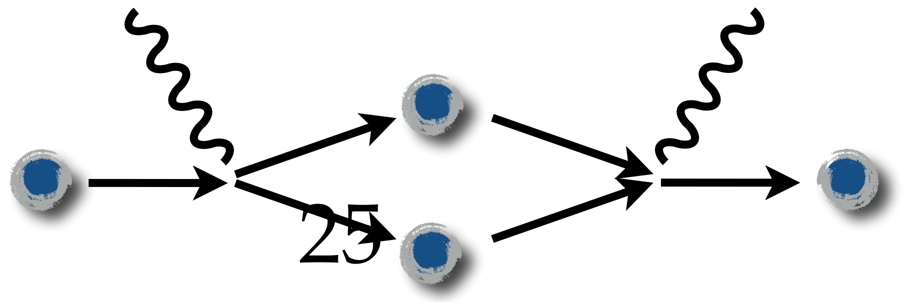


- ☑ Three-to-three scattering: $3 \to 3$



- ☑ Long distance matrix elements



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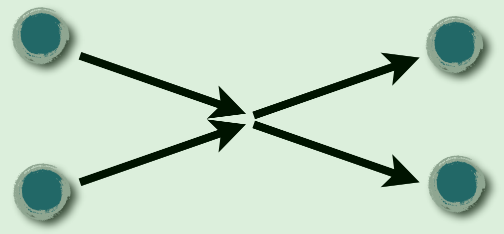

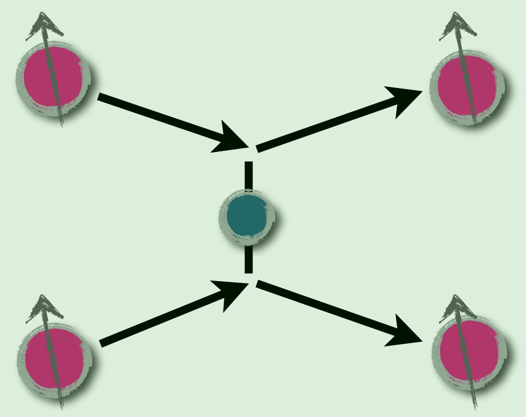

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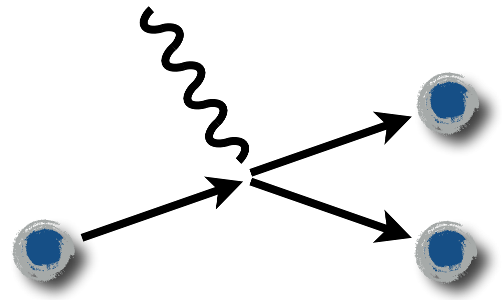

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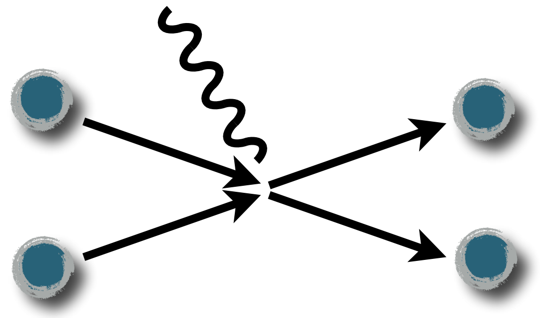

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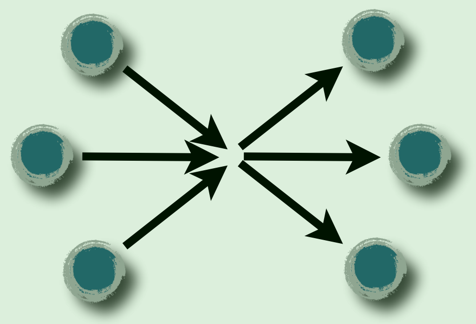

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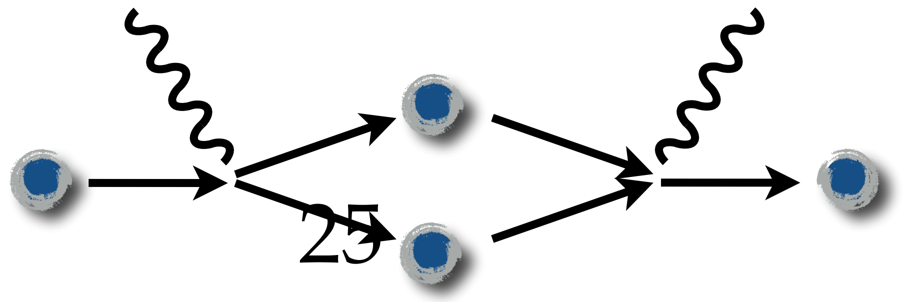

= *this talk*

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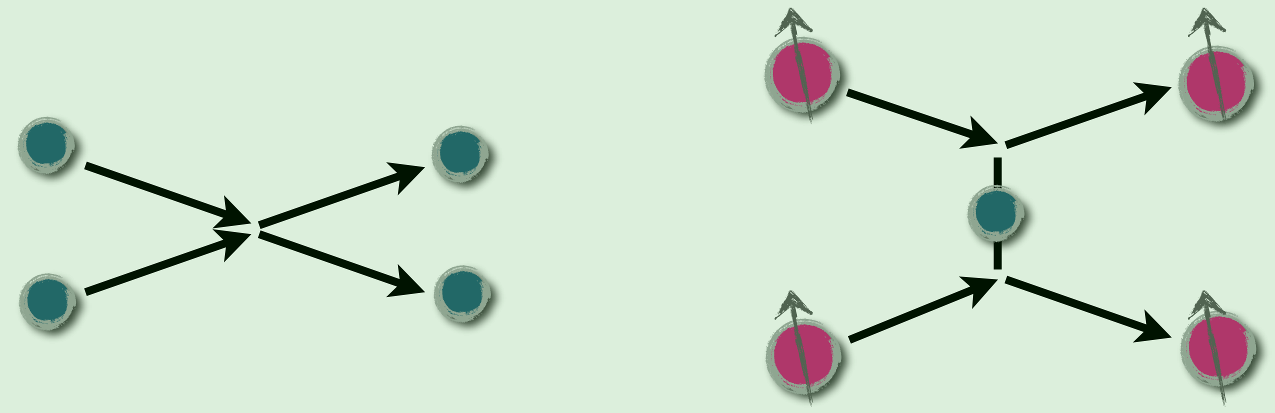

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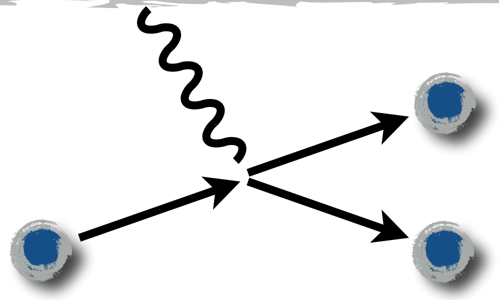

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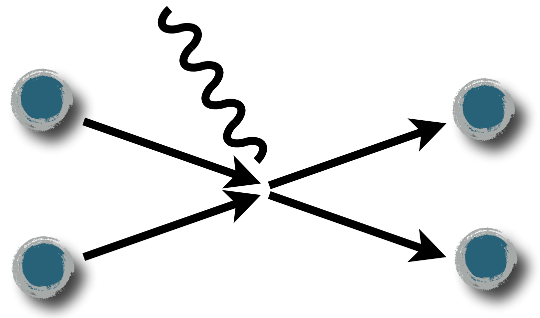

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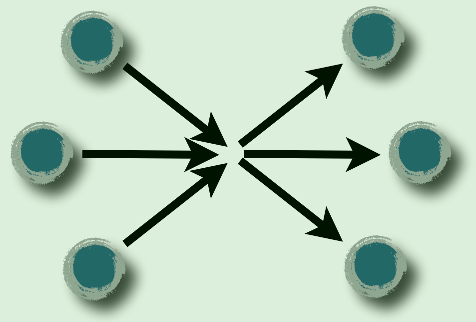

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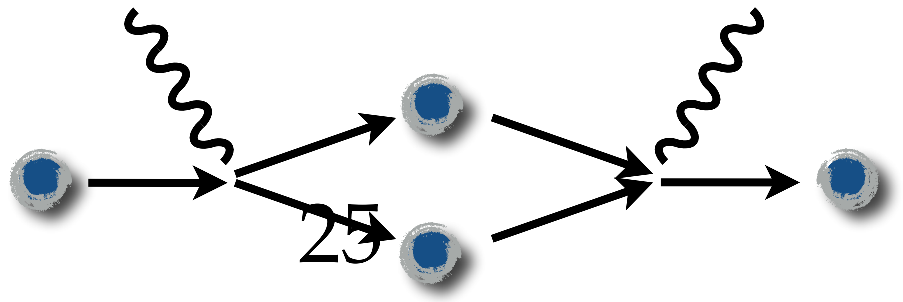
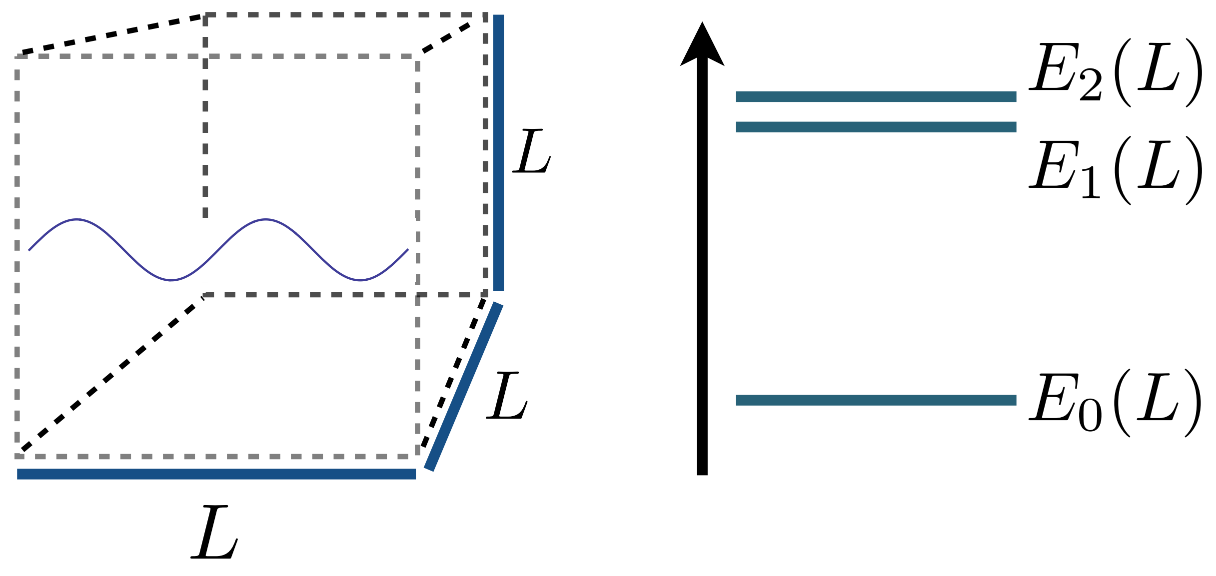


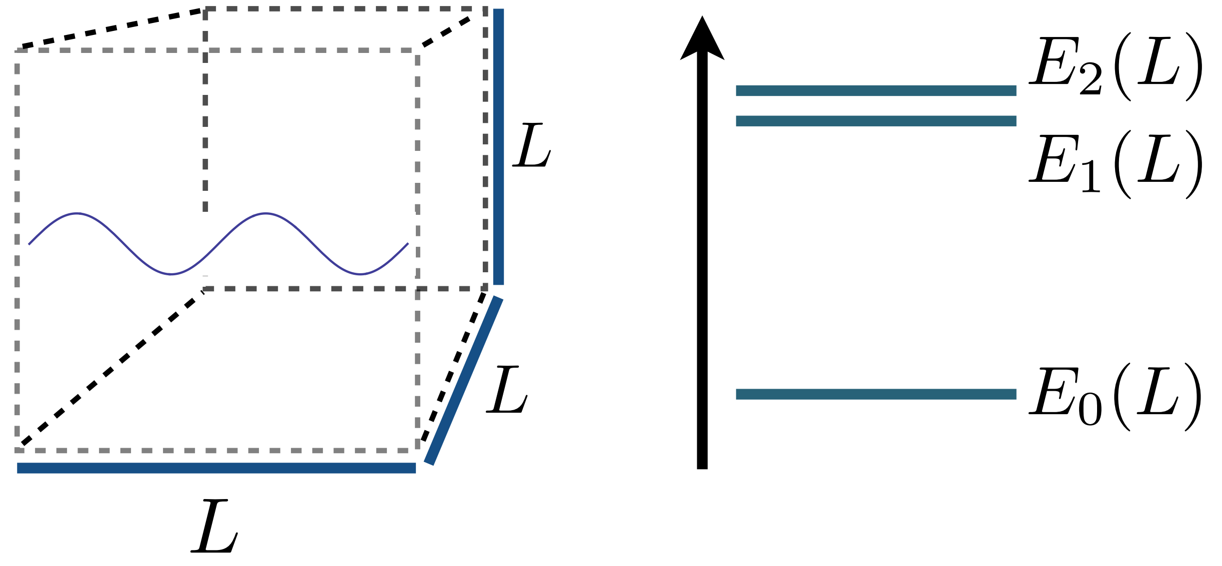
= *this talk*

The finite-volume



- Finite, cubic, periodic spatial volume (extent *L*)
- Discrete momenta and energies
- *L* is large enough to neglect $e^{-m_\pi L}$
- Finite *T* and non-zero lattice spacing assumed negligible

The finite-volume



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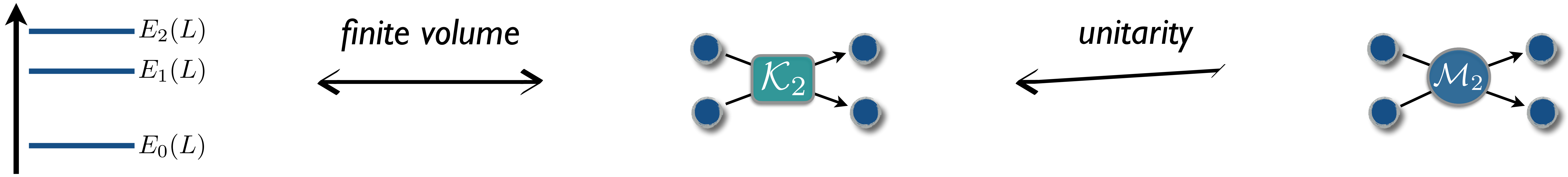
Quantization condition relates energies to K-matrices, and thereby amplitudes

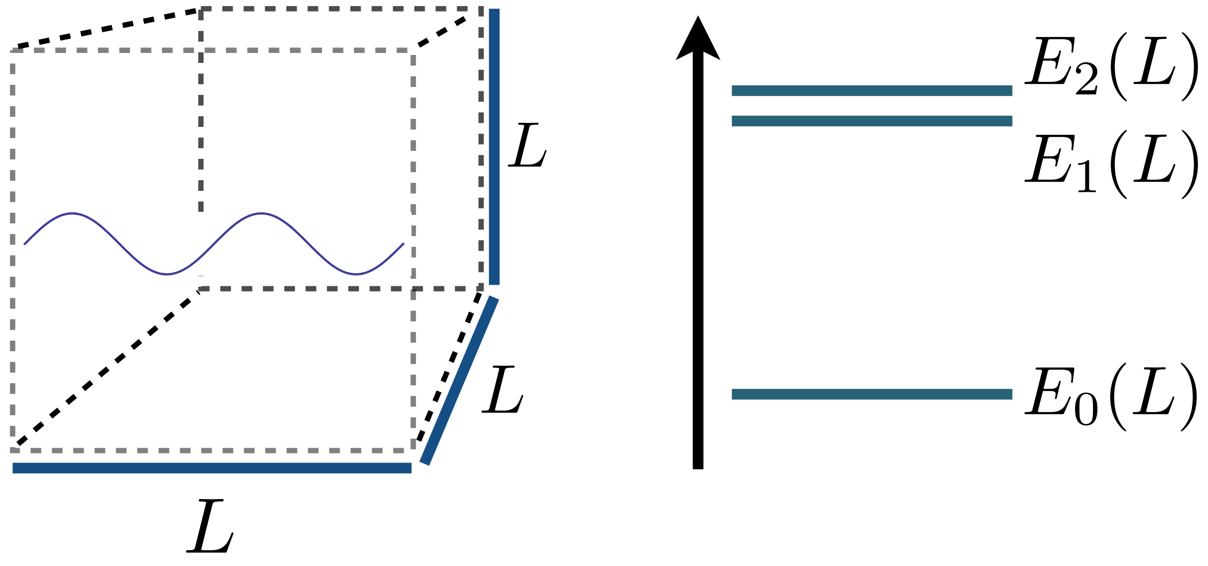


$$\det_{\ell m}[\mathcal{K}^{-1}(s) + F(P,L)] = 0$$

$F(P,L) \equiv$ Matrix of known geometric functions

• Lüscher (1989) • *many others* •

The finite-volume



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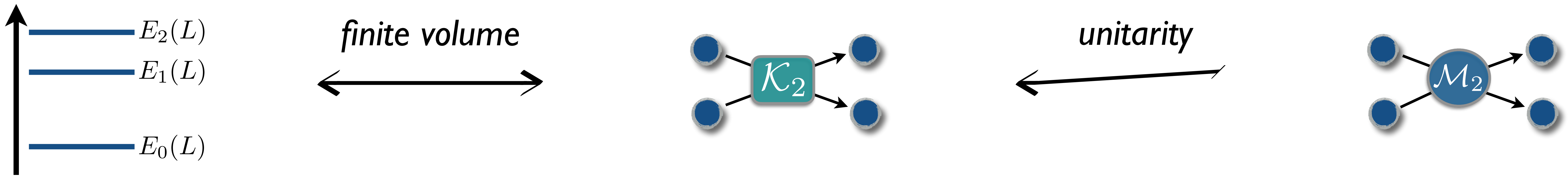

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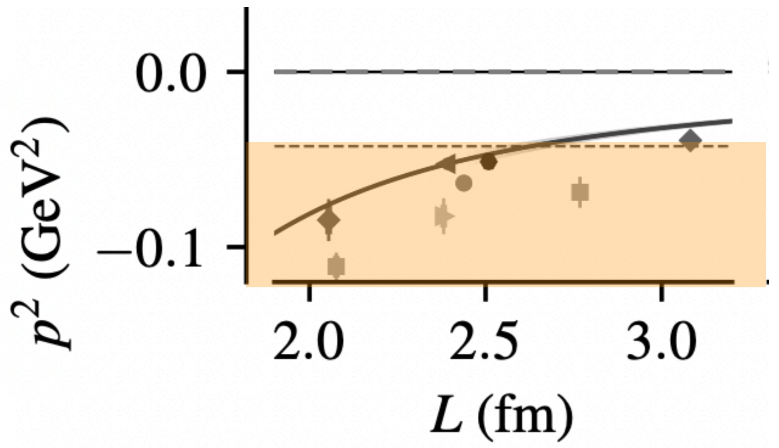
$F(P,L)$ is real below threshold

→ only solvable if $K(s)$ is real below threshold

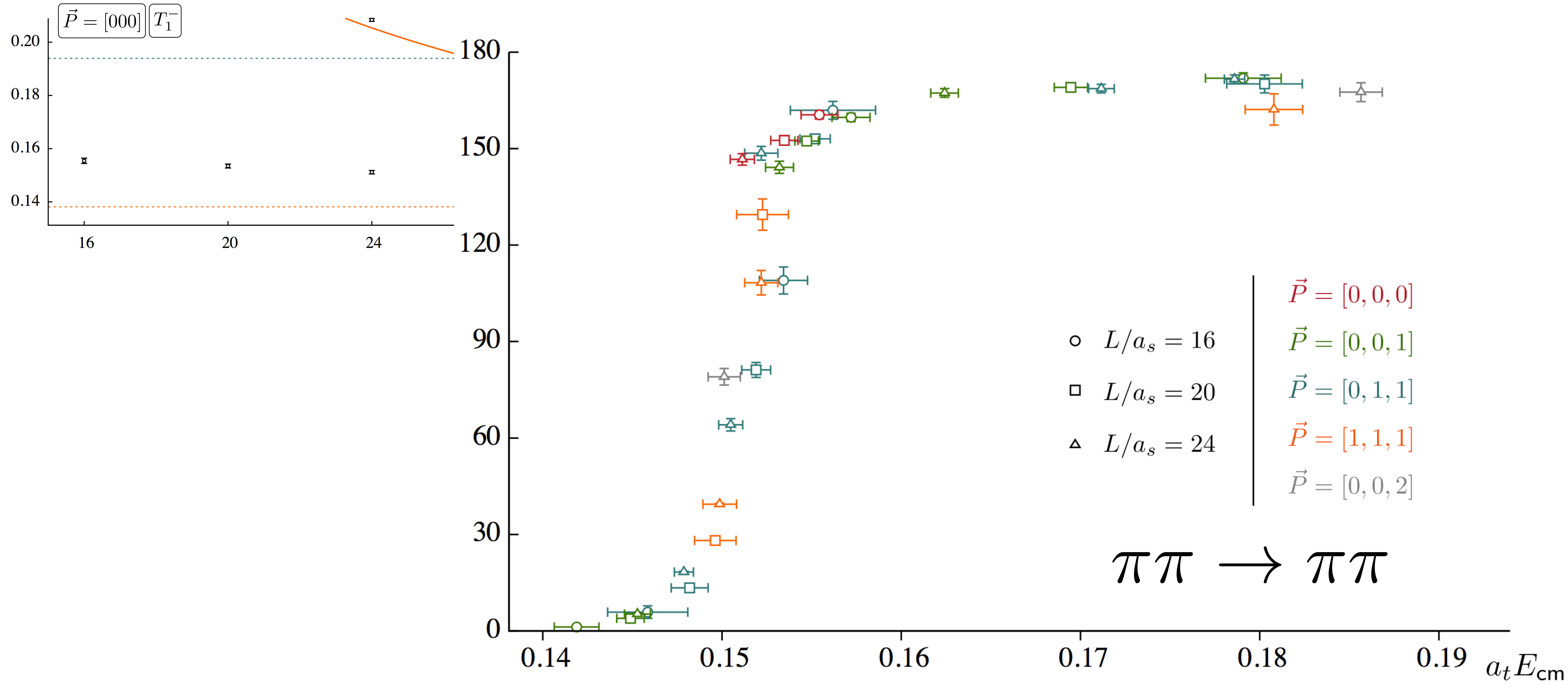


$$\frac{1}{s/4 - M_N^2} \log\left[\frac{s - 4M_N^2 + m_\pi^2 - i\epsilon}{m_\pi^2}\right]$$

Using the result (above threshold)

- Single-channel case *(pions in a p-wave)*

$$\mathcal{K}(s_n)^{-1} = \rho \cot\delta(s_n) = -F(E_n, \vec{P}, L)$$



• Dudek, Edwards, Thomas in *Phys.Rev*. D87 (2013) 034505 •

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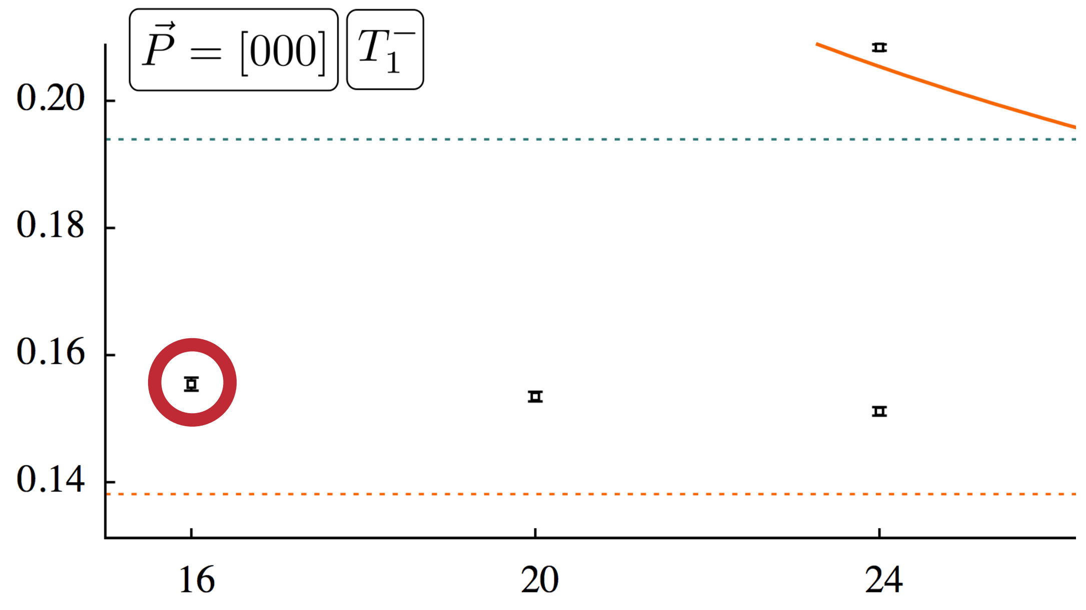

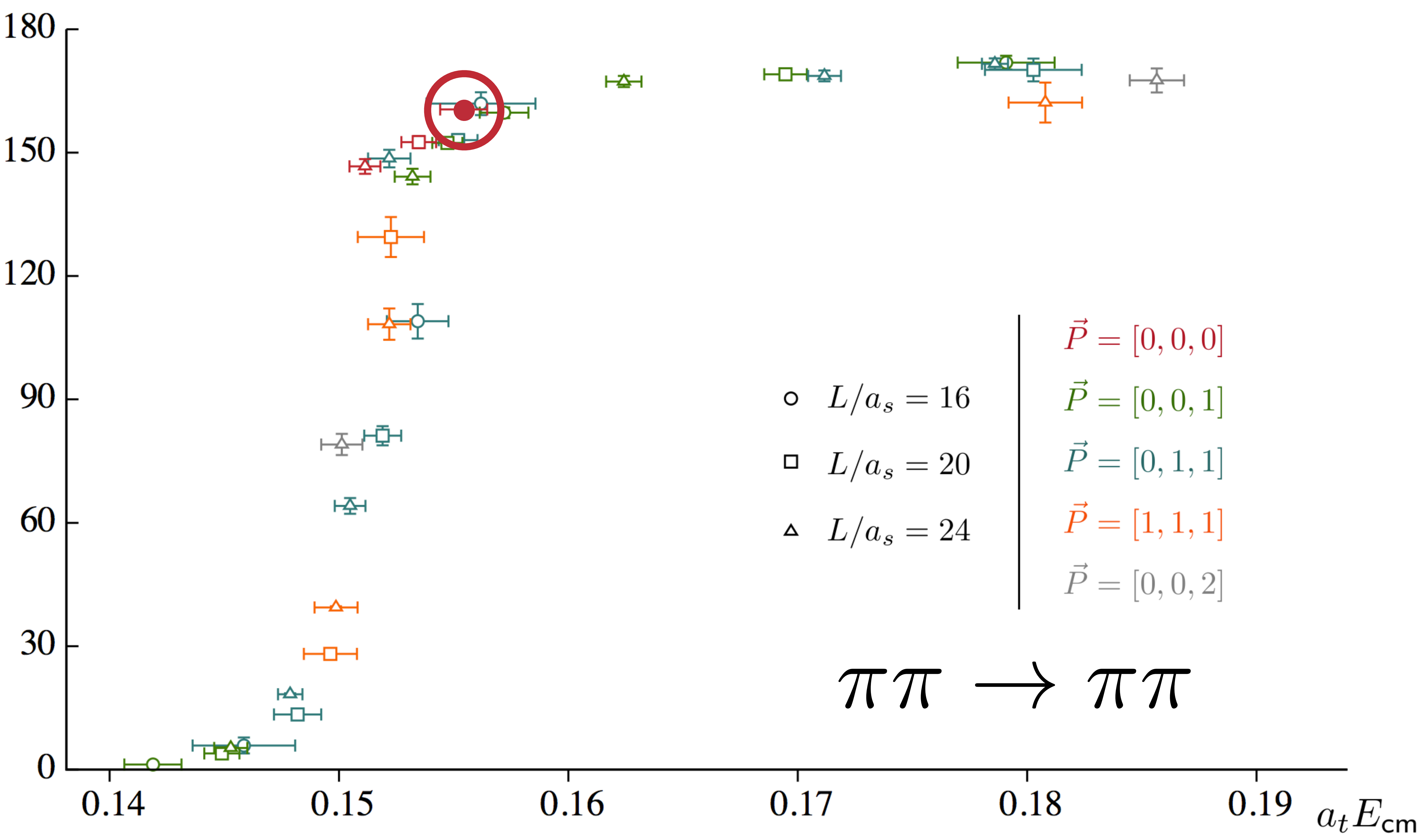

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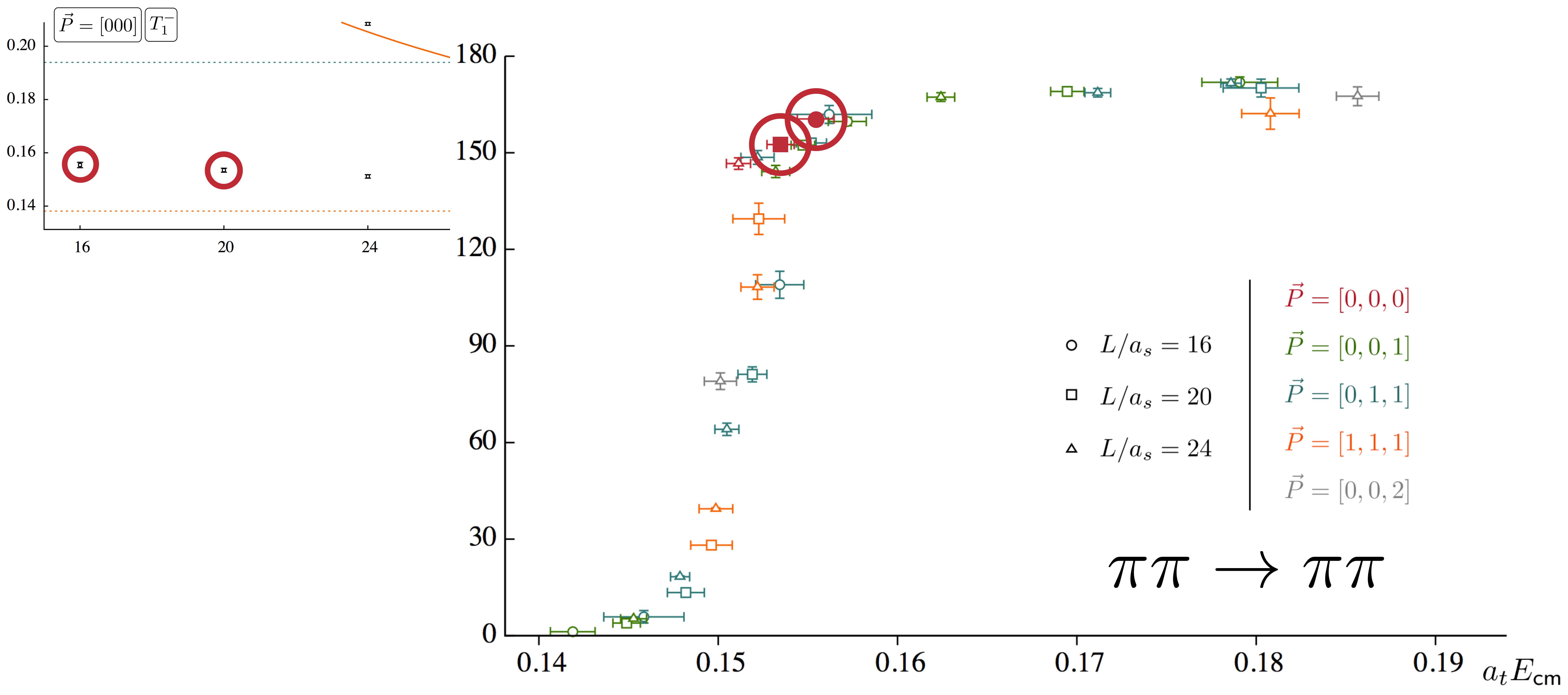

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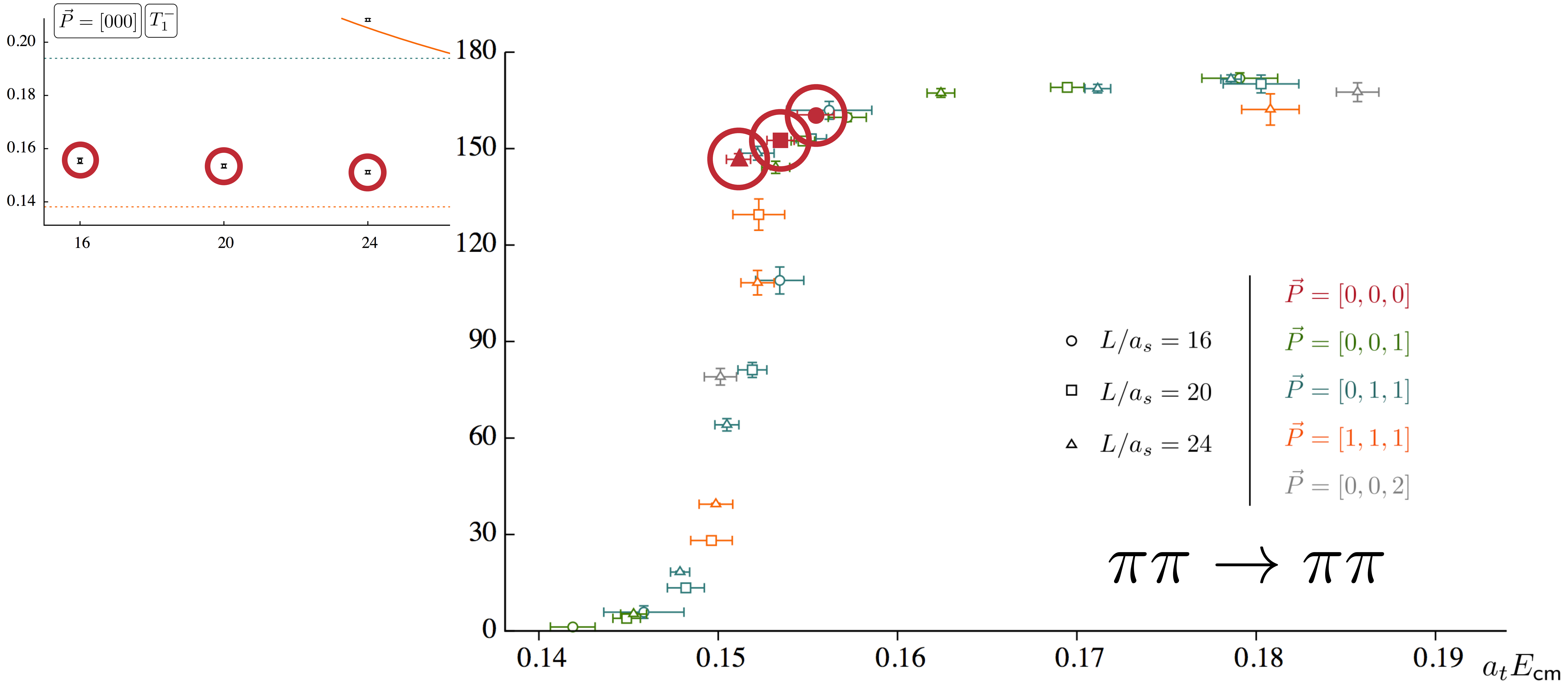

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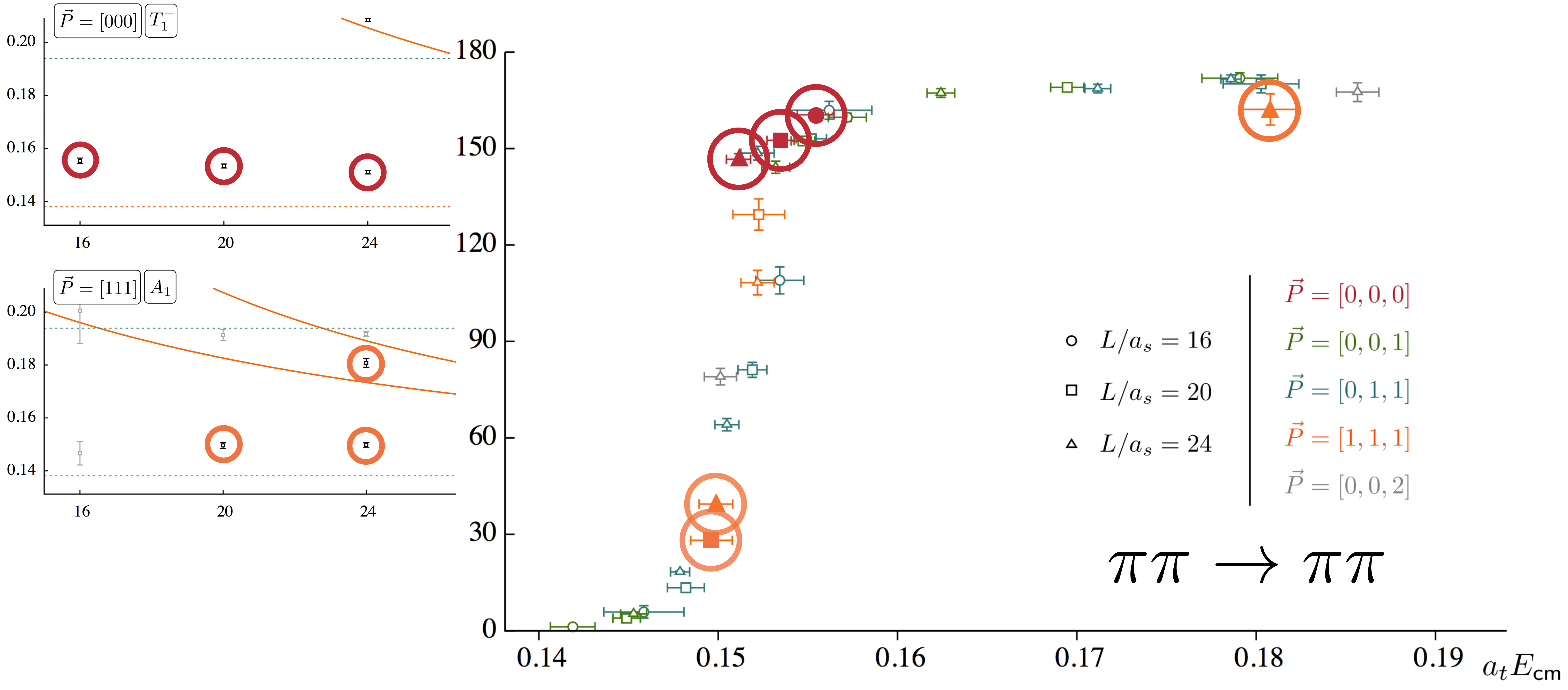


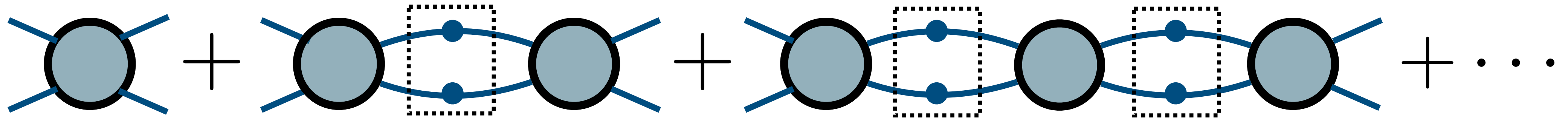
• Dudek, Edwards, Thomas in *Phys.Rev.* D87 (2013) 034505 •

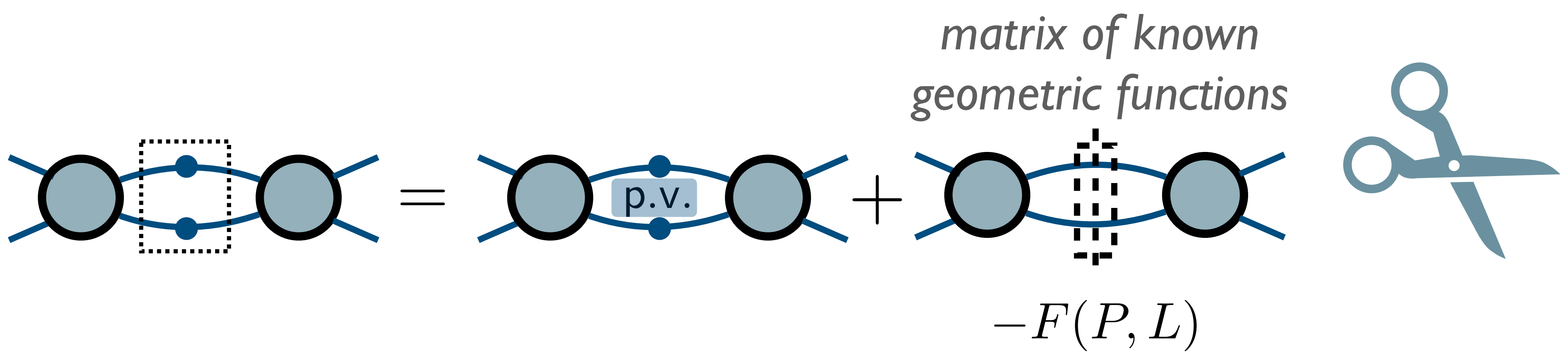
Back to the derivation

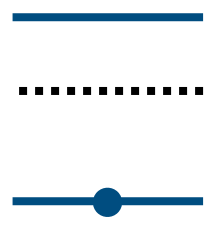
$\mathcal{M}_L(P) =$ finite-volume correlator

poles are finite-volume energies

$$\mathcal{M}_L(P) =$$

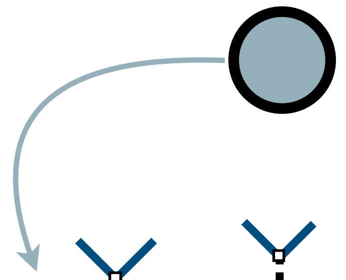




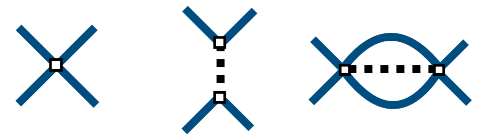


propagating hadrons

= fully dressed



Bethe-Salpeter kernel



$$= \sum_{\boldsymbol{k}} [\text{real, analytic}]$$

$$= \int_{\boldsymbol{k}} [\text{real, analytic}] + e^{-\mu L}$$

for $(2M_N)^2 < s < (2M_N + m_\pi)^2$

Framework for generic, EFT independent, all-orders diagrammatic relations

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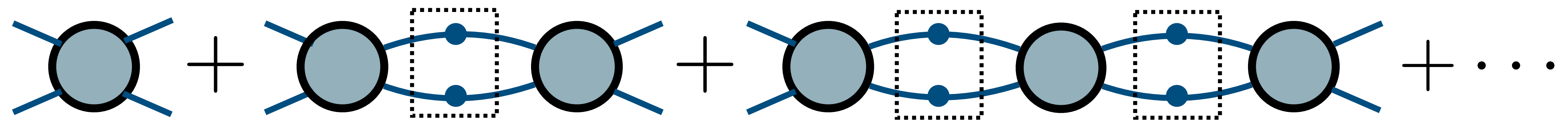
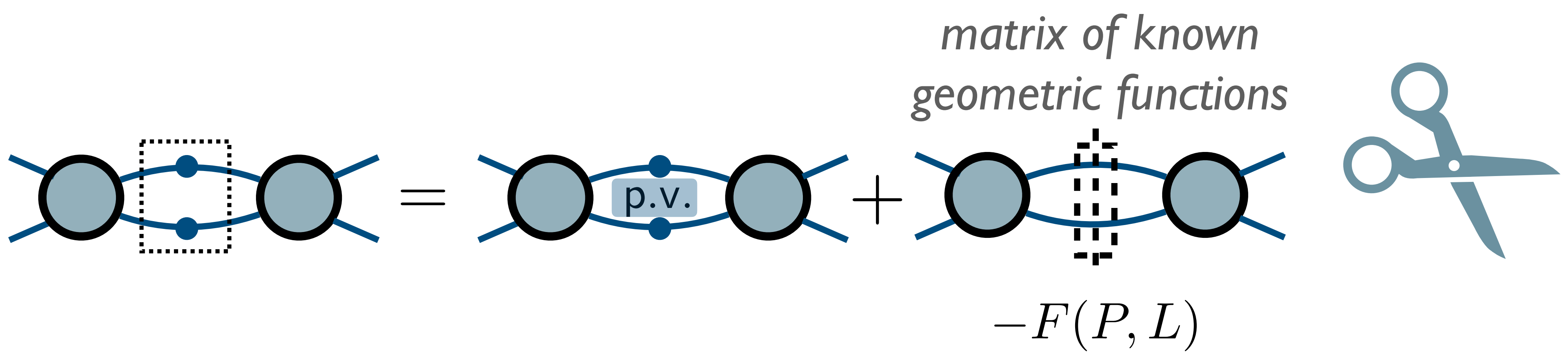
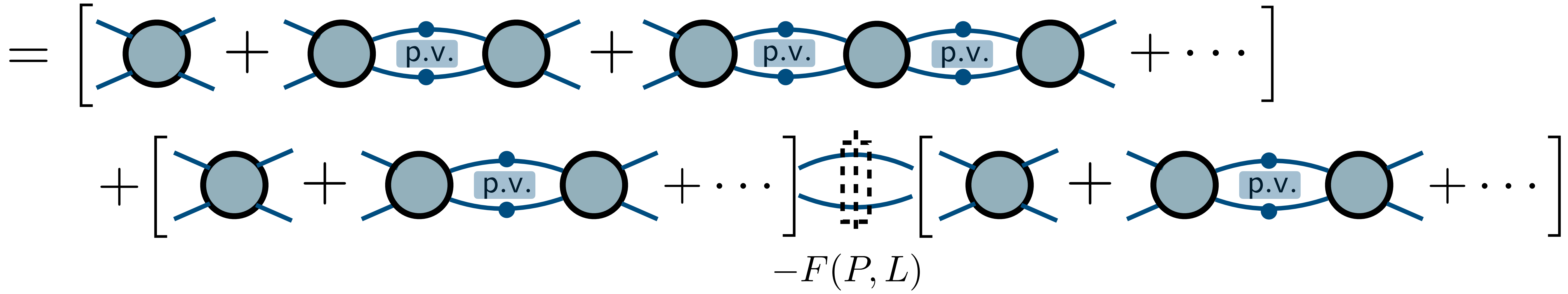
poles are finite-volume energies

$\mathcal{M}_L(P) =$

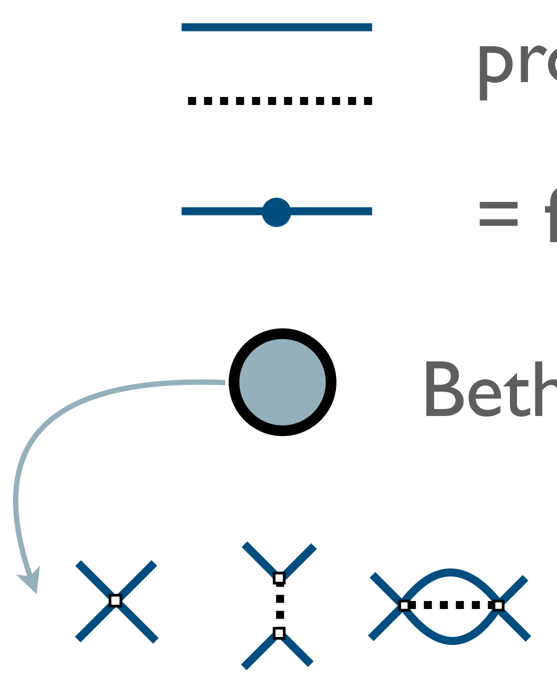




defines the K matrix $\mathcal{K}(s)$



$$= \mathcal{K}(s) - \mathcal{K}(s)F(P,L)\mathcal{K}(s) + \cdots = \frac{1}{\mathcal{K}(s)^{-1} + F(P,L)}$$



$= \sum_{\boldsymbol{k}}$ [real, analytic]

$= \int_{\boldsymbol{k}}$ [real, analytic] $+ e^{-\mu L}$

for $(2M_N)^2 < s < (2M_N + m_\pi)^2$

Framework for generic, EFT independent, all-orders diagrammatic relations

Key details:

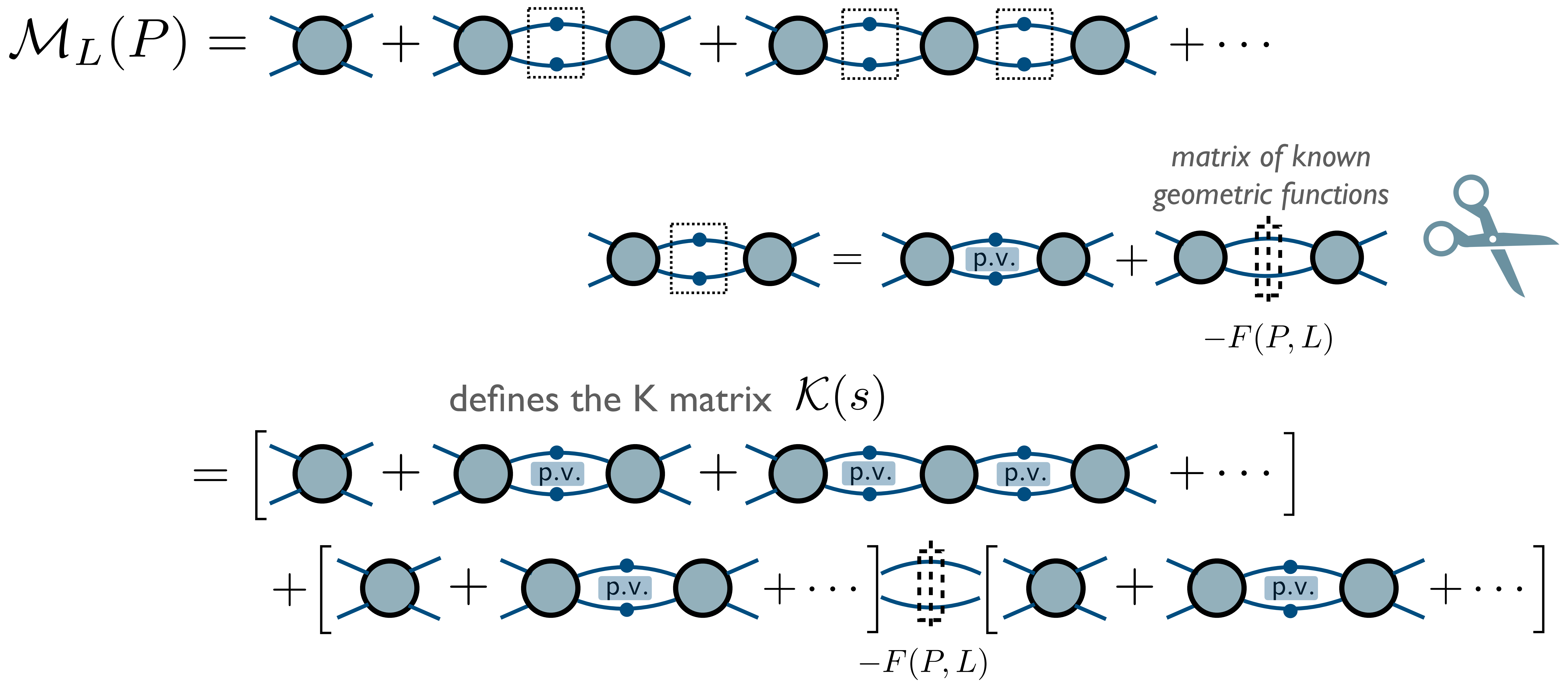
matrices on angular momentum $\otimes$ channels

$K(s)$ populated with physical (on-shell) partial waves

Back to the derivation

$\mathcal{M}_L(P) =$ finite-volume correlator

poles are finite-volume energies



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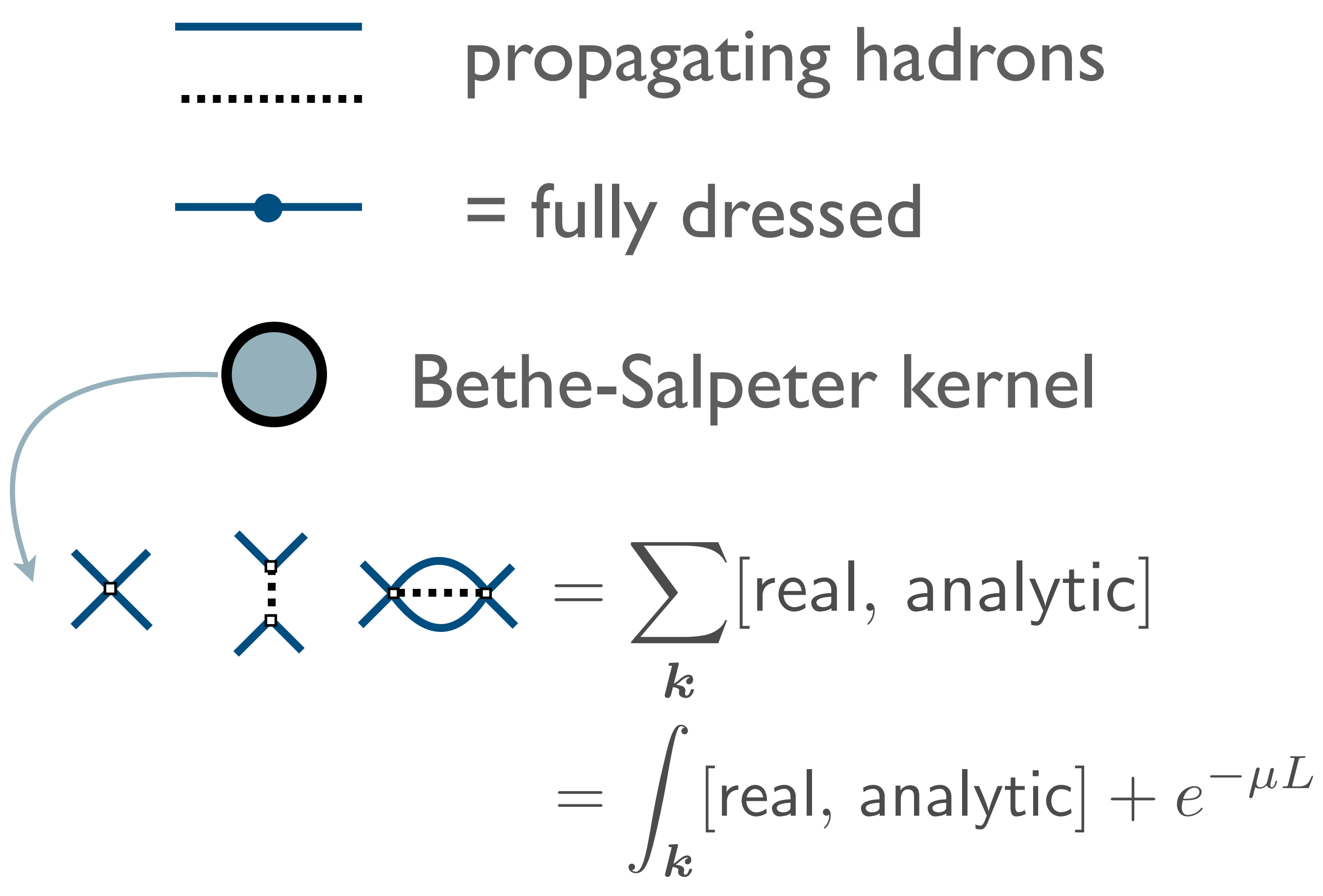


for $(2M_N)^2 < s < (2M_N + m_\pi)^2$

Framework for generic, EFT independent, all-orders diagrammatic relations

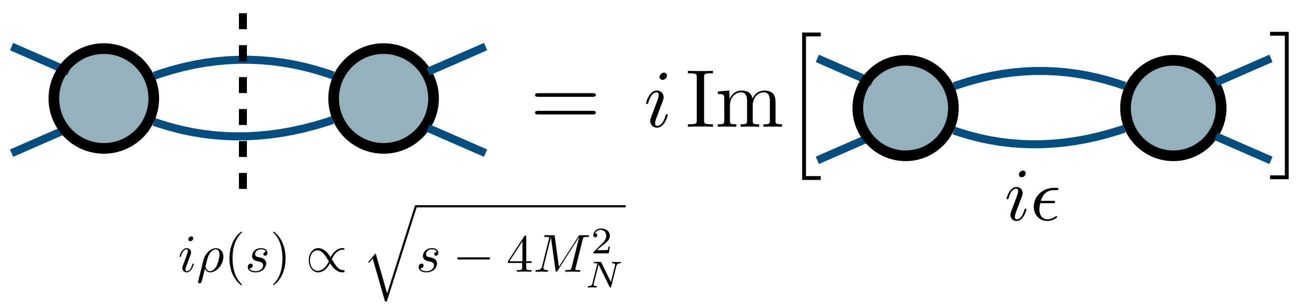
Key details:

matrices on angular momentum $\otimes$ channels

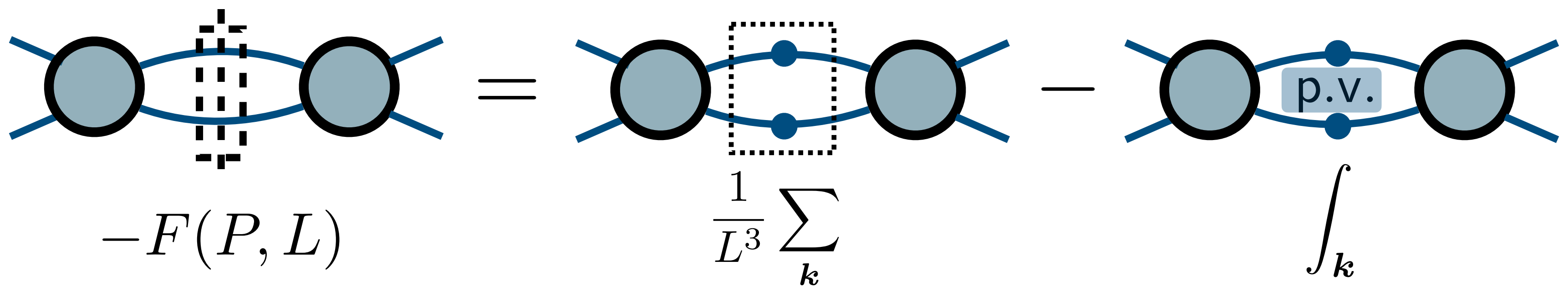
$K(s)$ populated with physical (on-shell) partial waves

What goes wrong with the left-hand cuts?

Origins / issues of on-shell projection



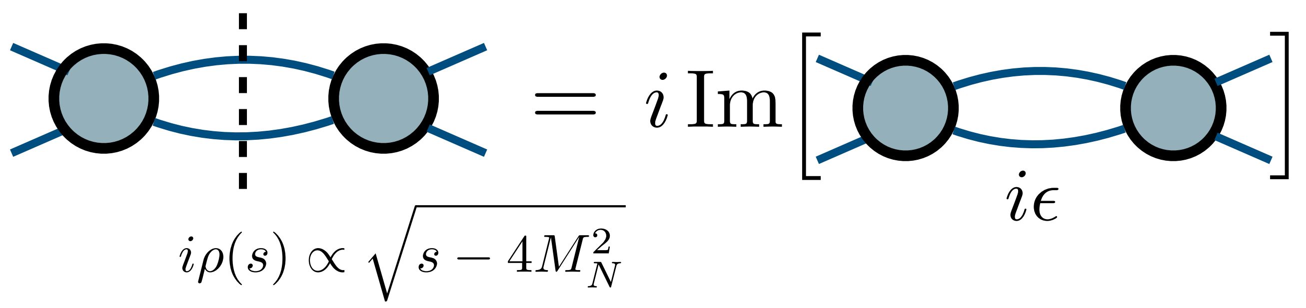
Exactly set on the mass shell by a Dirac delta function



Set on the mass shell by the relation:

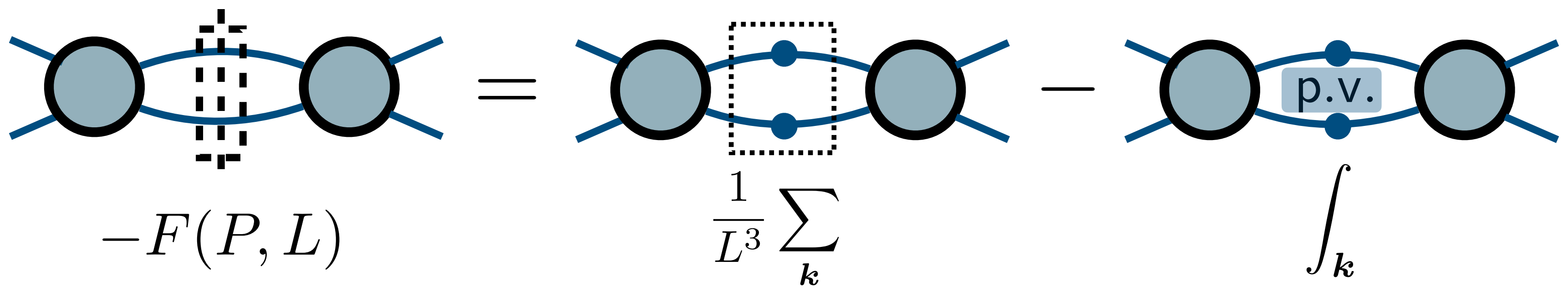
$$\left[\frac{1}{L^3}\sum_{\boldsymbol{k}} - \int_{\boldsymbol{k}}\right]\frac{1}{2\omega_{\boldsymbol{k}}}\frac{\mathcal{S}(\boldsymbol{k}_{\text{cm}}^2) - \mathcal{S}(p(E_{\text{cm}})^2)}{\boldsymbol{k}_{\text{cm}}^2 - p(E_{\text{cm}})^2} = e^{-\mu L}$$

Origins / issues of on-shell projection



Exactly set on the mass shell by a Dirac delta function

… a result that leads to the correct sub-threshold analytic continuation



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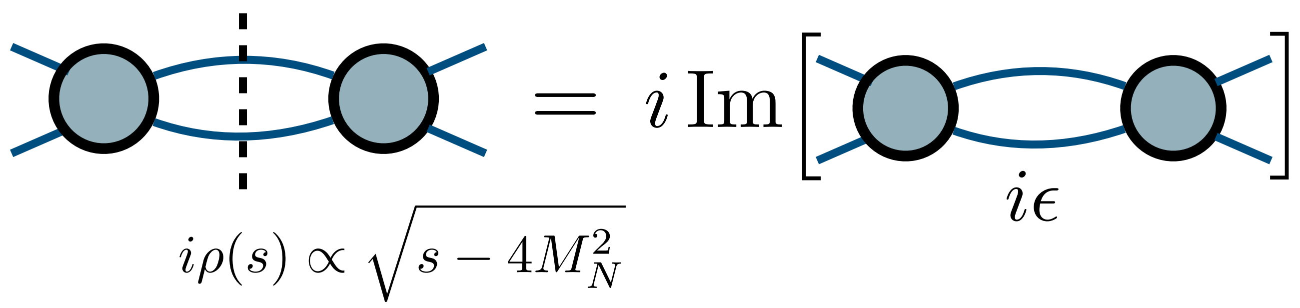
… a result that breaks on the cut

$$\frac{1}{\boldsymbol{k}_{\text{cm}}^2}\log\left[\frac{4\boldsymbol{k}_{\text{cm}}^2+m_\pi^2-i\epsilon}{m_\pi^2}\right] \qquad \frac{1}{s/4-M_N^2}\log\left[\frac{s-4M_N^2+m_\pi^2-i\epsilon}{m_\pi^2}\right]$$

Origins / issues of on-shell projection

MTH and Raposo, JHEP (2024)



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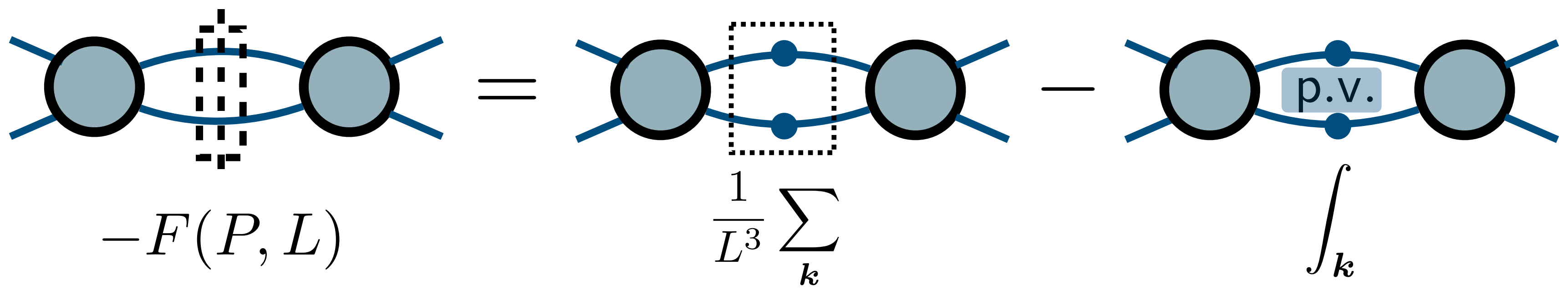

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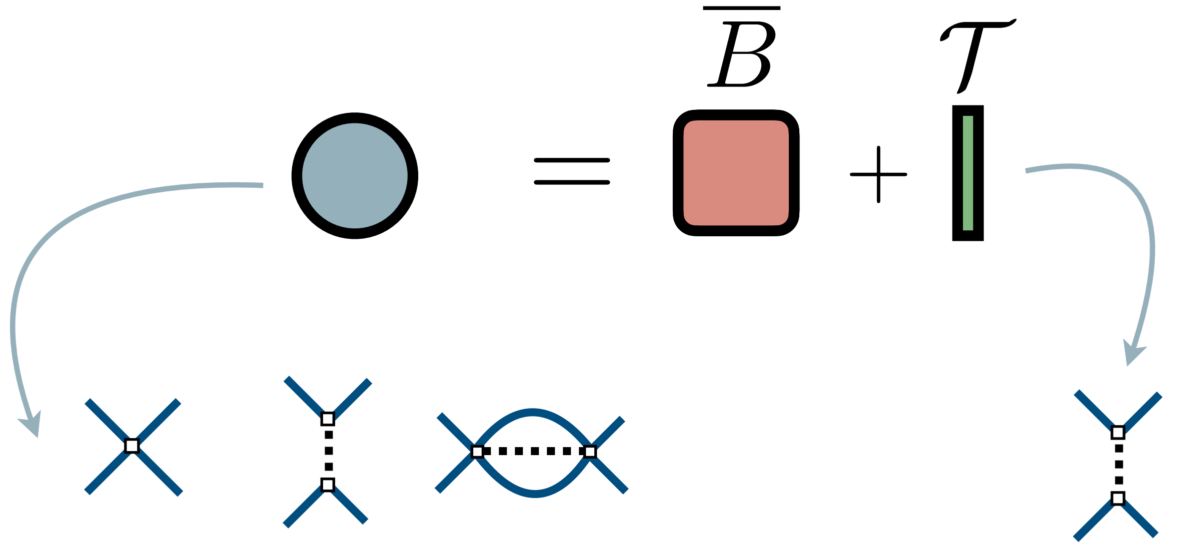
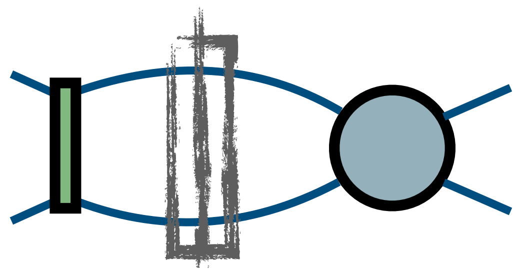
Resolution:

separate the t-channel exchange from the Bethe Salpeter kernel



change the details of the cut to keep T partly off-shell



modify index space to reach a new quantisation condition

$$\ell, m \longrightarrow |\boldsymbol{k}_{\text{cm}}|, \ell, m$$

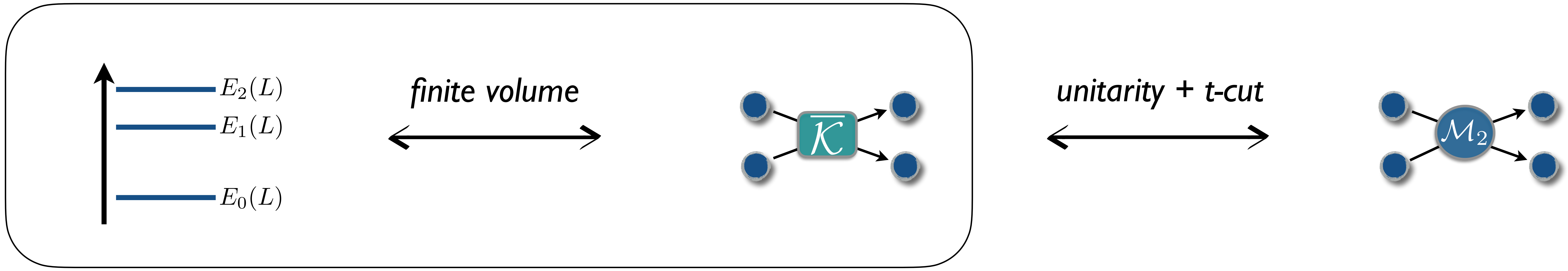
$E_n(L)$ can now constrain modified K matrix + the cut

$$\overline{\mathcal{K}}(s) \qquad g^2 \log[\cdots]$$

Known integral equations relate this to the standard K (or the amplitude)

New quantization condition…

MTH and Raposo, JHEP (2024)



$$\det_{\boldsymbol{k}_{\mathsf{cm}}\ell m}\left[S(P,L)^{-1}+\xi^\dagger\,\overline{\mathcal{K}}^{\mathsf{os}}(P)\,\xi+2g^2\mathcal{T}(P)\right]=0$$

$S(P,L) =$ Matrix of known geometric functions

$\mathcal{T} =$ Matrix of known off-shell logs

e.g. $\mathcal{T}_{|\boldsymbol{k}_{\mathsf{cm}}|00,|\boldsymbol{k}'_{\mathsf{cm}}|00}=-\dfrac{1}{|\boldsymbol{k}_{\mathsf{cm}}||\boldsymbol{k}'_{\mathsf{cm}}|}\log\left[\dfrac{2\omega_{\boldsymbol{k}_{\mathsf{cm}}}\omega_{\boldsymbol{k}'_{\mathsf{cm}}}-2|\boldsymbol{k}_{\mathsf{cm}}||\boldsymbol{k}'_{\mathsf{cm}}|-2M_N^2+m_\pi^2-i\epsilon}{2\omega_{\boldsymbol{k}_{\mathsf{cm}}}\omega_{\boldsymbol{k}'_{\mathsf{cm}}}+2|\boldsymbol{k}_{\mathsf{cm}}||\boldsymbol{k}'_{\mathsf{cm}}|-2M_N^2+m_\pi^2-i\epsilon}\right]$

$\xi_{\boldsymbol{k}_{\mathsf{cm}}}=1 \quad \longleftrightarrow \quad \xi=\begin{pmatrix}1 & 1 & \cdots & 1 & 1\end{pmatrix}$

$g = NN\pi$ coupling

$\overline{\mathcal{K}}^{\mathsf{os}}(P) =$ Diagonal matrix of Lorentz scalars

New matrix space, truncated by cutoff function

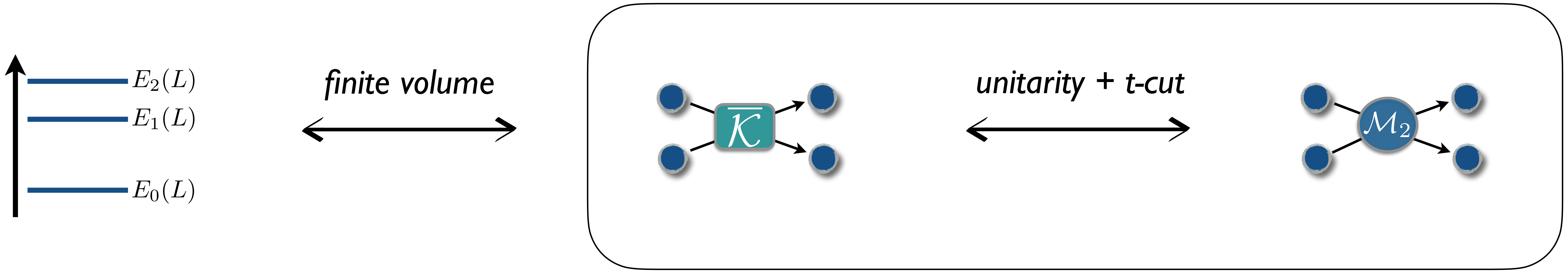
Holds up to $e^{-m_\pi L}$

Inspired in part three-particle work…

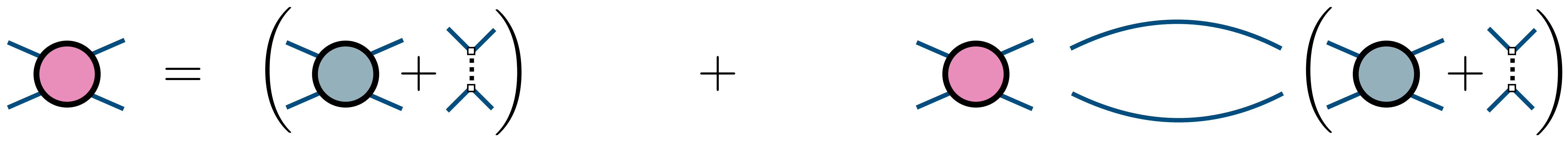
Blanton, Briceño, Döring, Draper, Mai, Meißner, Müller, Hammer, MTH, Pang, Romero-López, Rusetsky, Sharpe

New integral equations…

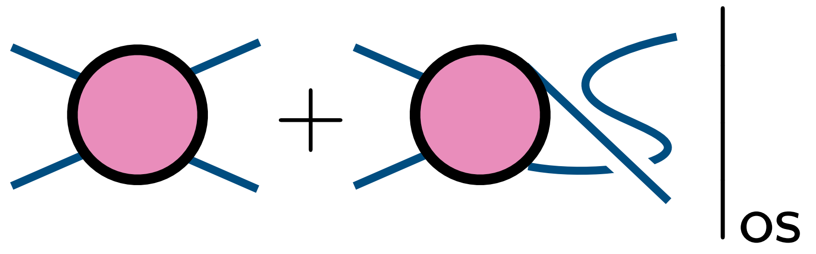
MTH and Raposo, JHEP (2024)



$$\mathcal{M}^{\mathsf{aux}}(P,p,p') = \mathcal{K}^{\mathcal{T}}(P,p,p') - \frac{1}{2}\int \frac{d^3\boldsymbol{k}^\star}{(2\pi)^3}\,\frac{\mathcal{M}^{\mathsf{aux}}(P,p,k)\,H(\boldsymbol{k}^\star)\,\mathcal{K}^{\mathcal{T}}(P,k,p')}{4\omega(\boldsymbol{k}^\star)\left[(k^\star_{\mathsf{os}})^2 - (\boldsymbol{k}^\star)^2 + i\epsilon\right]}$$



$$\mathcal{K}^{\mathcal{T}}(P,p,p') = \overline{\mathcal{K}}^{\mathsf{os}}(P,p,p') + 2g^2\mathcal{T}(P,p,p')$$

$\mathcal{M}(s,t) =$ 

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Comparison to other methods…

- Sato and Bedaque: explicitly study the on-shell off-shell difference above the cut

$$\left[\frac{1}{L^3}\sum_{\boldsymbol{k}} - \int_{\boldsymbol{k}}\right]\frac{1}{2\omega_{\boldsymbol{k}}}\frac{\mathcal{S}(\boldsymbol{k}_{\text{cm}}^2) - \mathcal{S}(p(E_{\text{cm}})^2)}{\boldsymbol{k}_{\text{cm}}^2 - p(E_{\text{cm}})^2} = e^{-\mu L} \qquad \frac{1}{\boldsymbol{k}_{\text{cm}}^2}\log\left[\frac{4\boldsymbol{k}_{\text{cm}}^2 + m_\pi^2 - i\epsilon}{m_\pi^2}\right] \qquad \frac{1}{s/4 - M_N^2}\log\left[\frac{s - 4M_N^2 + m_\pi^2 - i\epsilon}{m_\pi^2}\right]$$

- Meng and Epelbaum: different basis could mean a different truncation… *numerical checks needed*

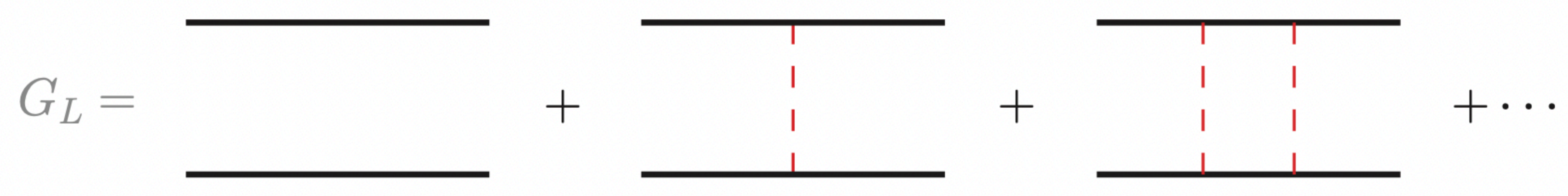
- Bubna et al.: perhaps the most similar, *I still need to understand the details*

Images taken from Akaki Rusetsky's Lattice2024 talk…

$$V(r) = \underbrace{V_L(r)}_{\text{known, local}} + \underbrace{V_S(r)}_{\text{unknown}}$$

$$T = T_L + (1 + T_L G_0) T_S (1 + G_0 T_L)$$

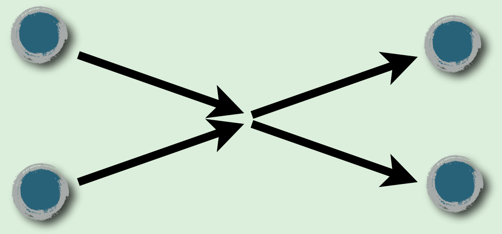
$$T_S = V_S + V_S G_L T_S \qquad G_L = G_0 + G_0 V_L G_L$$

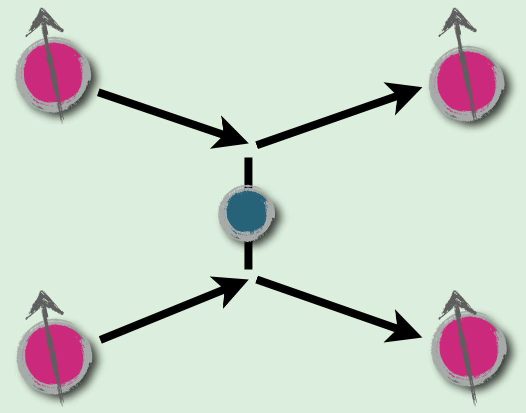


Landscape of amplitudes

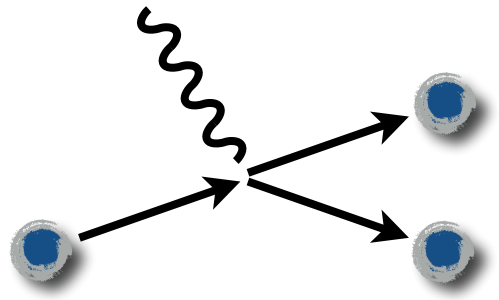
- Large body of work dedicated to extracting amplitudes from finite-volume data

- ☑ Two-to-two scattering: $2 \to 2$

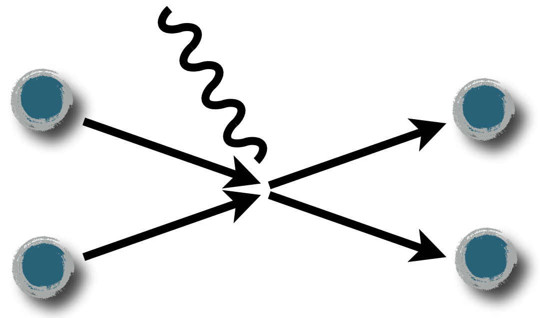




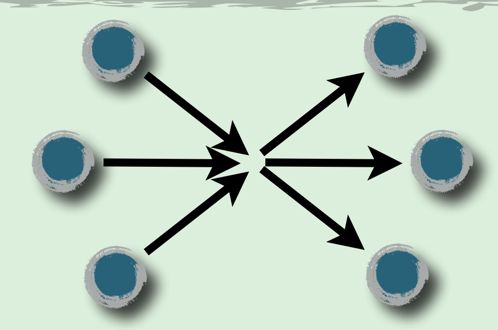
- ☑ Decays with an external current: $1 \xrightarrow[\mathcal{J}]{} 2$



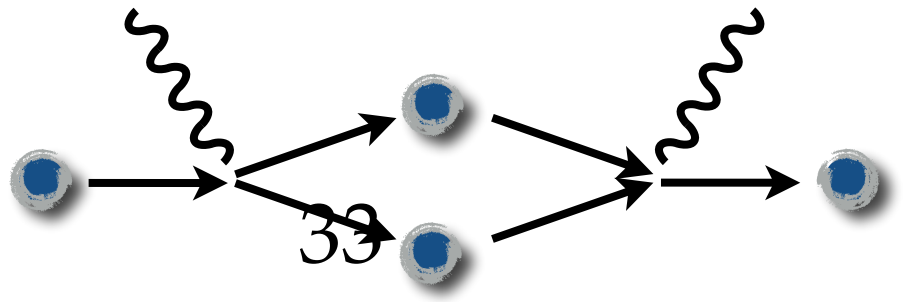
- ☑ Transitions with an external current: $2 \xrightarrow[\mathcal{J}]{} 2$



- ☑ Three-to-three scattering: $3 \to 3$



- ☑ Long distance matrix elements



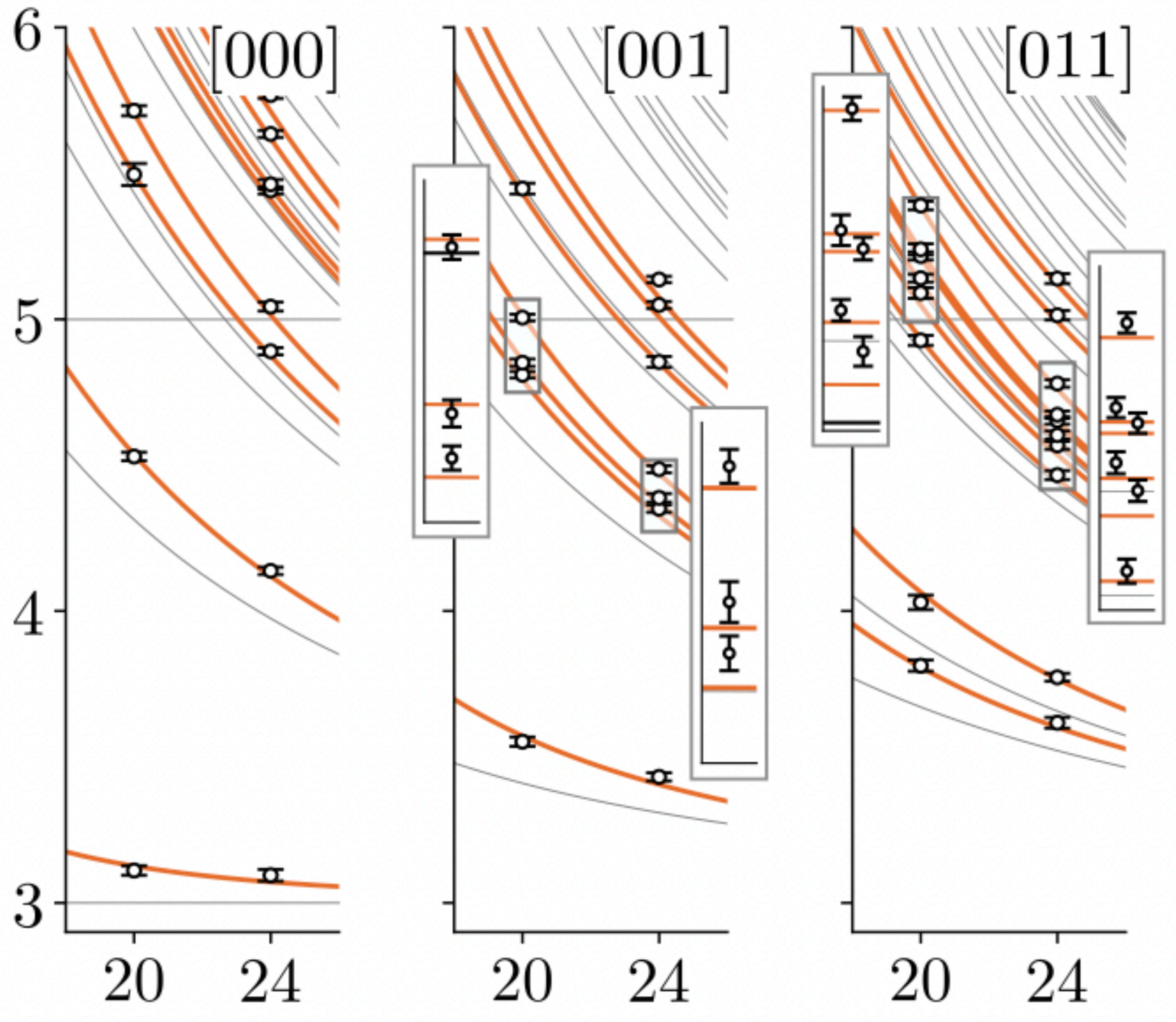
= *this talk*

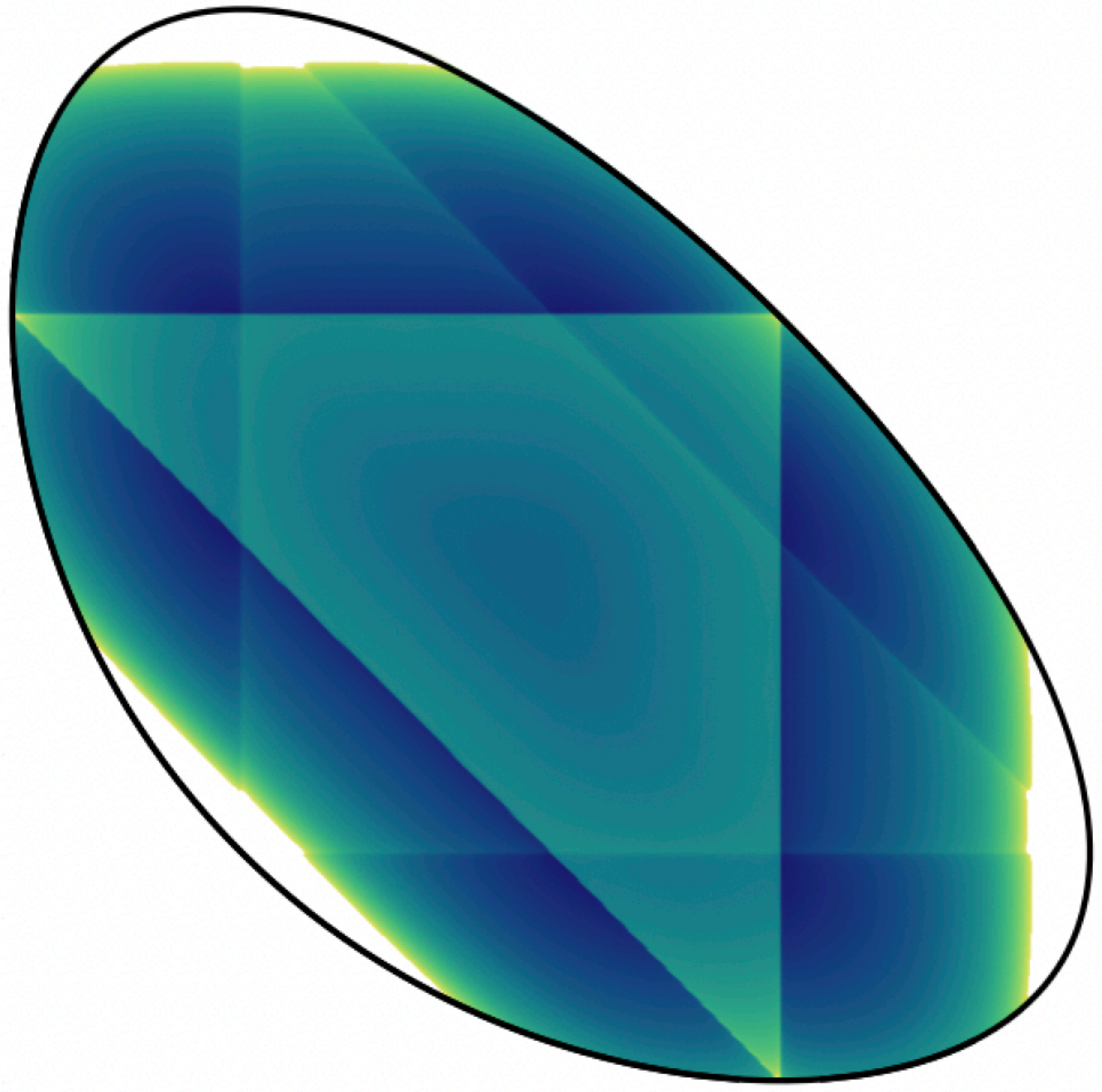
Towards >2 hadrons

- Multiple three-particle finite-volume formalisms developed

  MTH, Sharpe (2014-2016) *See also Döring, Mai, Hammer, Pang, Rusetsky*

- First lattice calculations appearing… e.g. $\pi^+\pi^+\pi^+ \to \pi^+\pi^+\pi^+$





- Extract reliable spectrum
- Use formalism to fit scheme-dependent K-matrix
- Solve integral equations to reach physical amplitude

MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

Towards >2 hadrons

Three-hadron dynamics from lattice QCD
Fernando Romero-Lopez
Tuesday, 9:40 - 10:20

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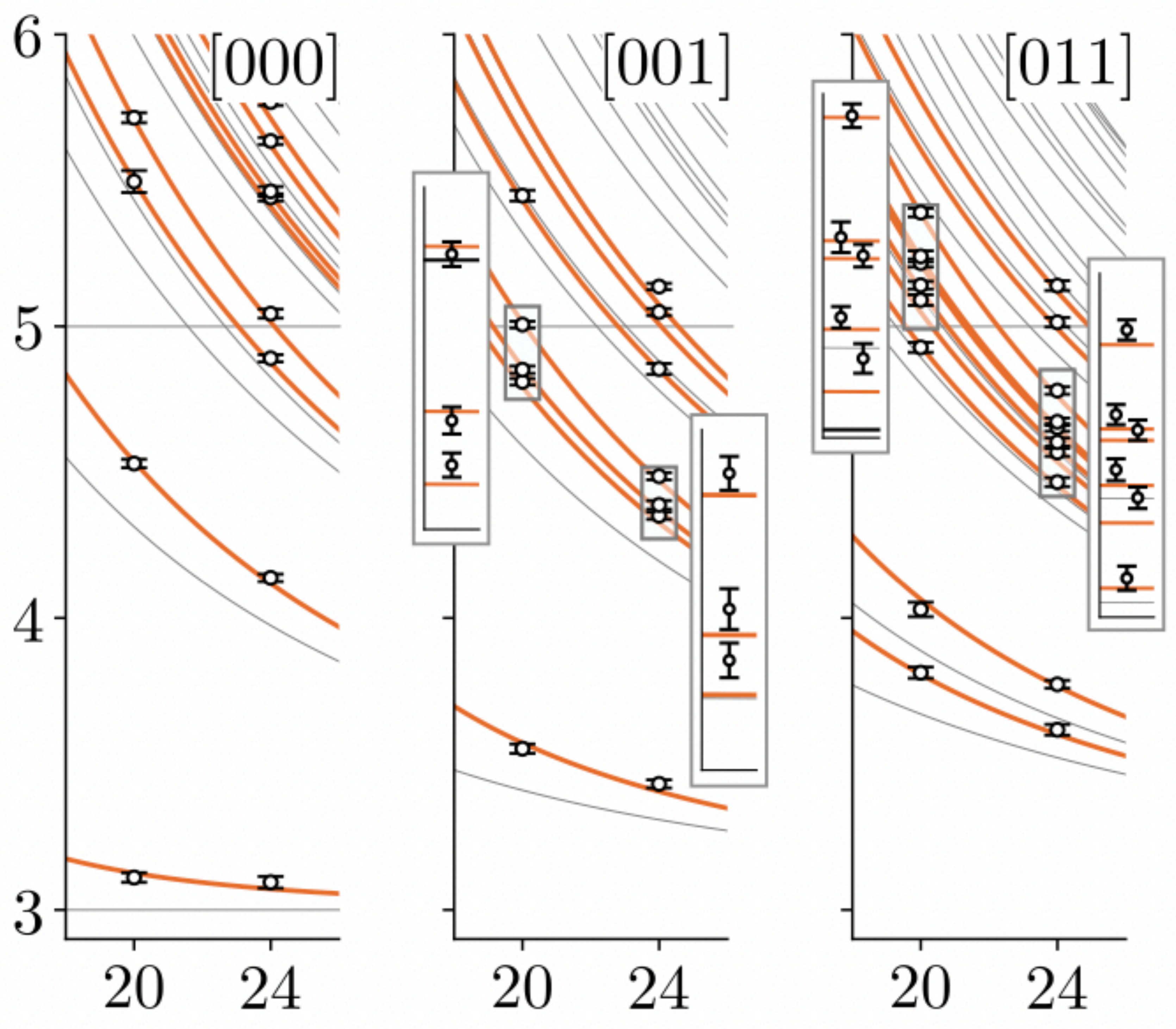

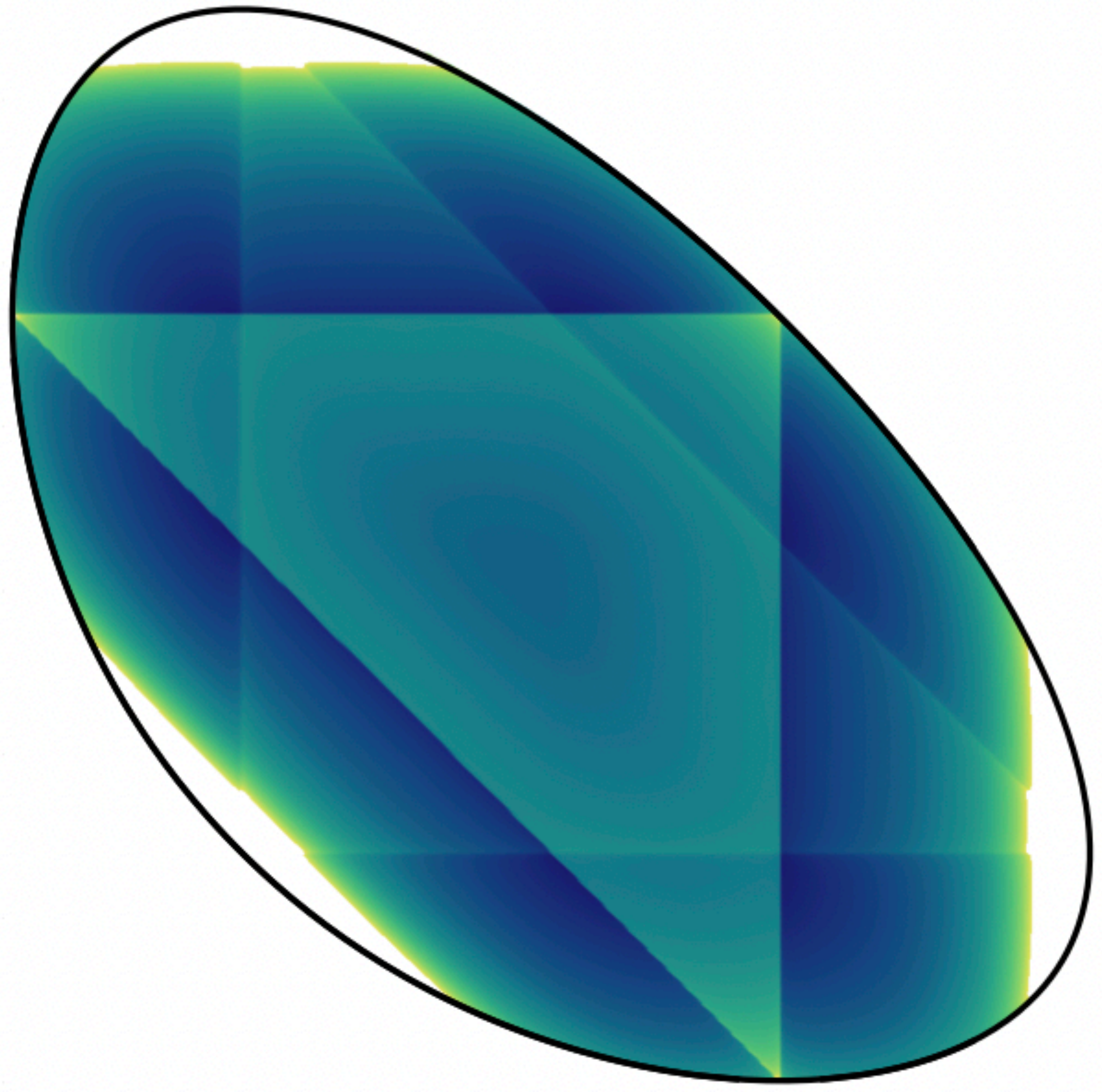

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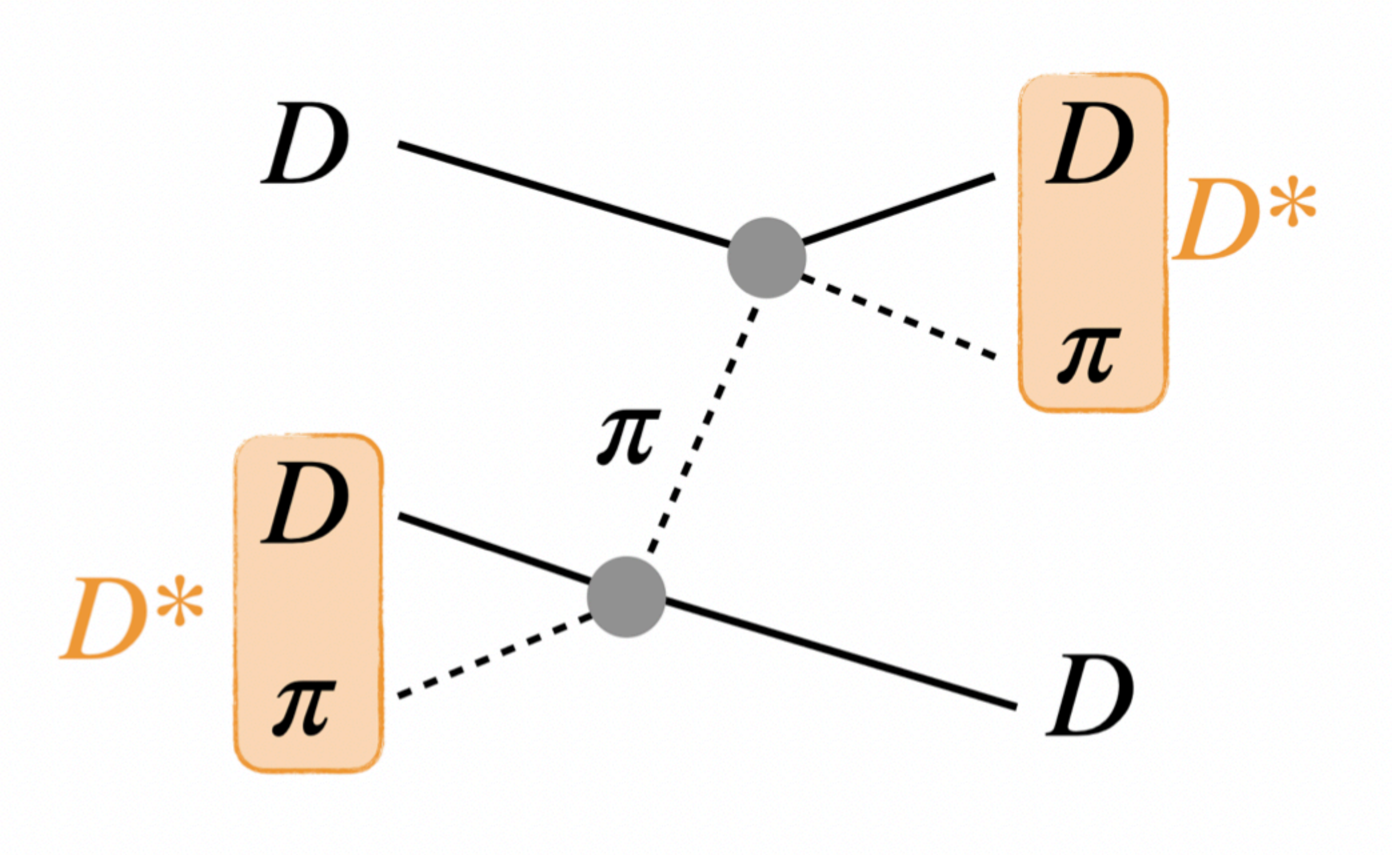
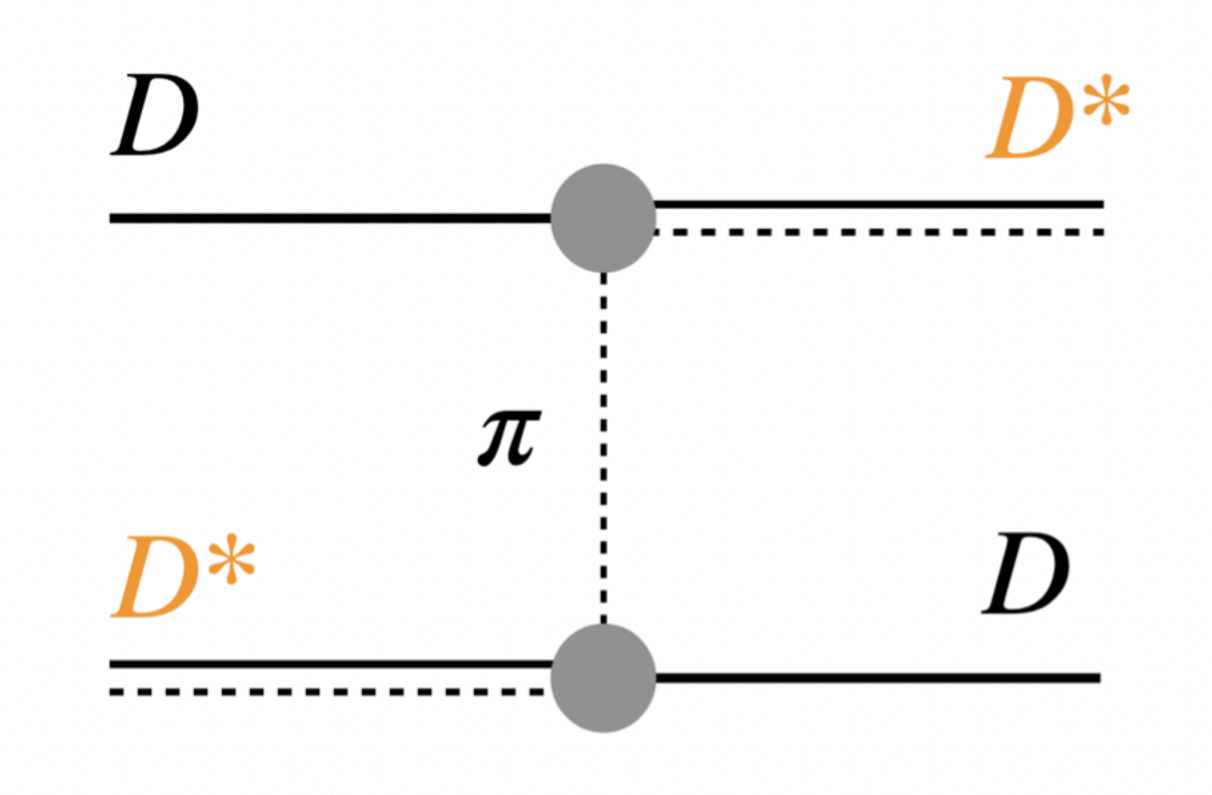
Three-body formula for left-hand cuts

Three-hadron dynamics from lattice QCD
Fernando Romero-Lopez
Tuesday, 9:40 - 10:20

- Three-body framework automatically includes left-hand cuts





- We recently extended the three-body framework to $DD^* \to DD\pi$

MTH, Romero-López and Sharpe, PRD (2024)

Summary and outlook

- Nearest left-hand cut is an [on-shell] + [angular-momentum projection] + [infinite-volume effect]

- [os] + [ang.-mom. proj.] + [∞-vol] Bethe-Salpeter kernel gives an incorrect description of the finite-volume system

- A modified derivation solves this at the expense of a new quantization condition (and a new K-matrix)

- Integrals relate new K-matrix to the scattering amplitude with the cut included

- Much to explore… finite-L effects near (but not on) the cut, connection to three-particle formalism

Thanks to Jeremy Green, Steve Sharpe, Fernando Romero-Lopez, Raúl Briceño for discussions!

Summary and outlook

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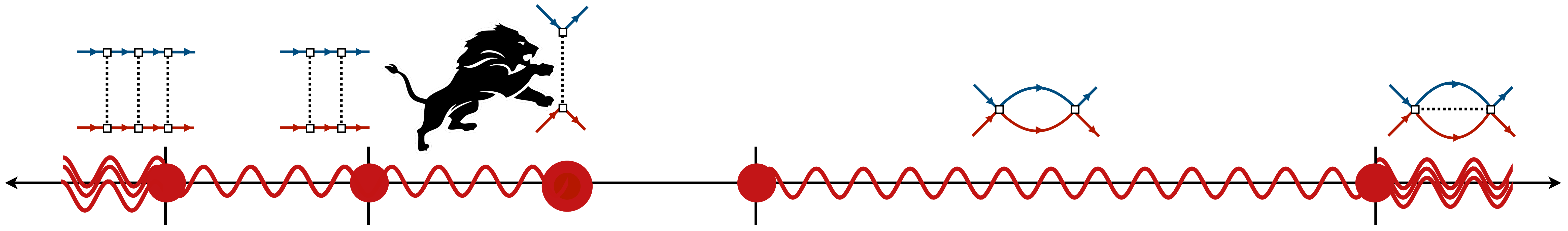


Thanks for listening! Questions?….

UK Research and Innovation

…back-up slides…

Recovering the standard formula

- Setting the coupling to zero gives

$$\det_{\boldsymbol{k}_{\mathrm{cm}}\ell m}\left[S(P,L)^{-1} + \xi^\dagger\,\overline{\mathcal{K}}^{\mathrm{os}}(P)\,\xi\right] = 0 \quad \longleftrightarrow \quad \det_{\boldsymbol{k}_{\mathrm{cm}}\ell m}\left[\overline{\mathcal{K}}^{\mathrm{os}}(P)^{-1} + \xi S(P,L)\xi^\dagger\right] = 0$$

- The contracted *S*-factor is closely related to *F*

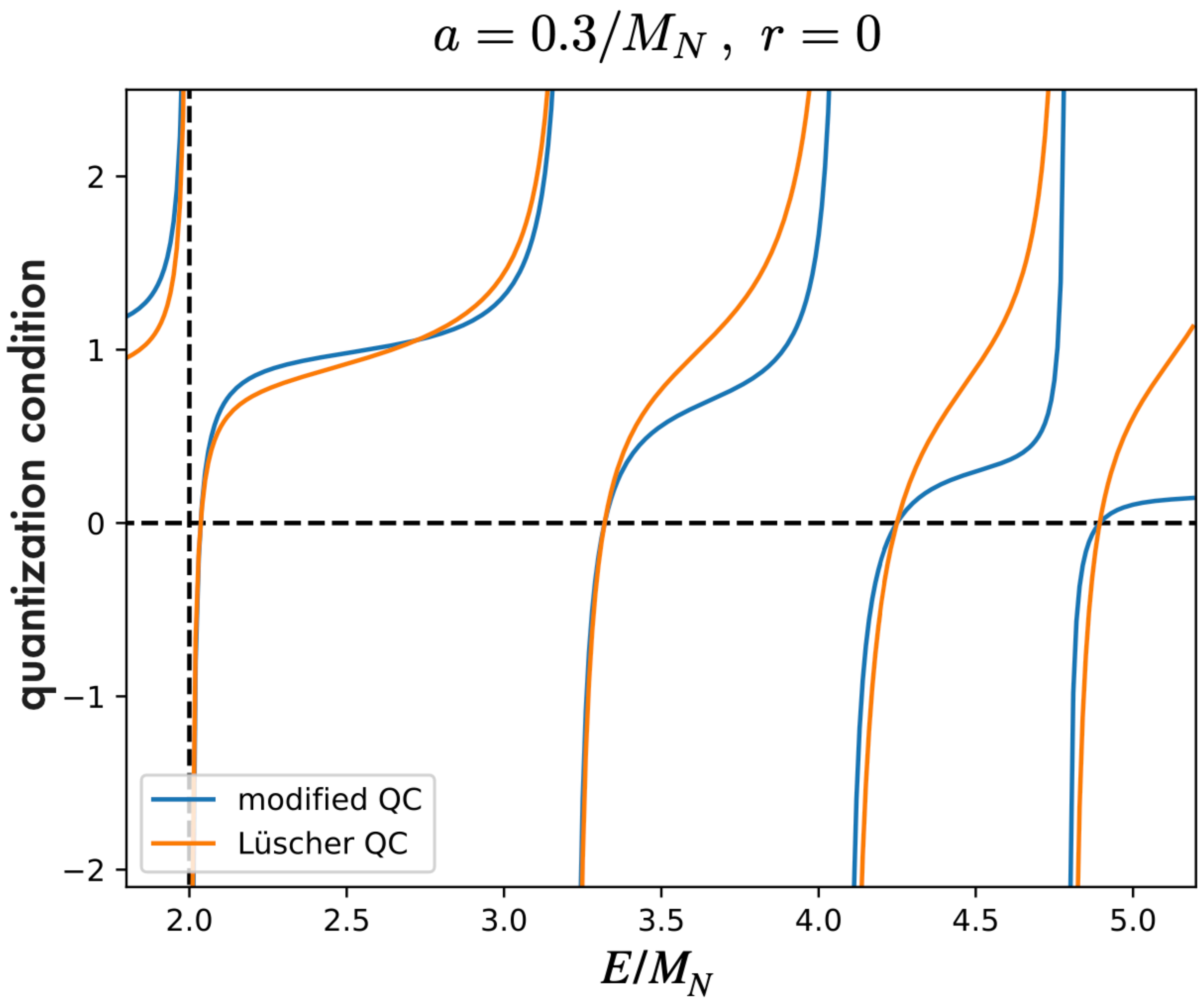
$$\xi S(P,L)\xi^\dagger = F(P,L) + I(P)$$

$$I_{\ell m,\ell' m'}(P) = \text{p.v.}\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{1}{2}\frac{4\pi\, Y_{\ell m}(\hat{\boldsymbol{k}}^\star)\, Y^*_{\ell' m'}(\hat{\boldsymbol{k}}^\star)\, |\boldsymbol{k}^\star|^{\ell+\ell'}\, H(\boldsymbol{k}^\star)}{2\omega_N(\boldsymbol{k})\, 2\omega_N(\boldsymbol{P}-\boldsymbol{k})\,(E-\omega_N(\boldsymbol{k})-\omega_N(\boldsymbol{P}-\boldsymbol{k}))}$$

- Integral equations give

$$\overline{\mathcal{K}}^{\mathrm{os}}(P)^{-1} = \mathcal{M}(P)^{-1} - I^{i\epsilon}(P) \qquad \mathcal{K}(P)^{-1} = \mathcal{M}(P)^{-1} + i\rho(s)$$

$$\overline{\mathcal{K}}^{\mathrm{os}}(P)^{-1} = \mathcal{K}(P)^{-1} - I(P)$$



- Put it together

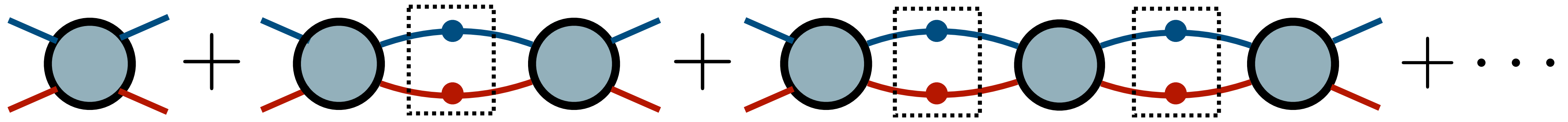
$$\det_{\ell m}\left[\mathcal{K}(P_j)^{-1} - I(P_j) + F(P_j,L) + I(P_j)\right] = \det_{\ell m}\left[\mathcal{K}(P_j)^{-1} + F(P_j,L)\right] = 0$$

Back to the derivation

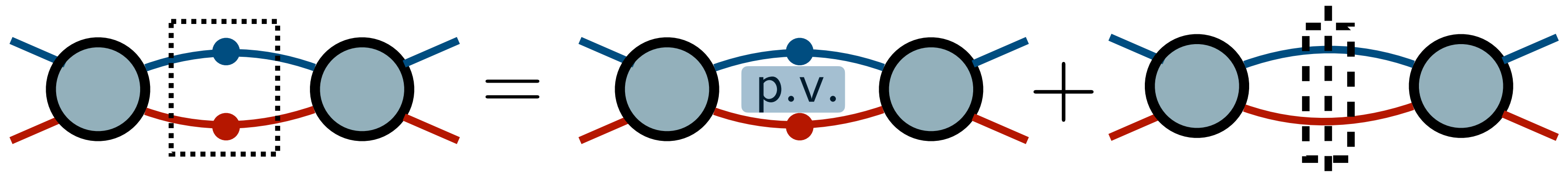
$\mathcal{M}_L(P) =$ finite-volume correlator

poles are finite-volume energies

$$\mathcal{M}_L(P) =$$



matrix of known geometric functions



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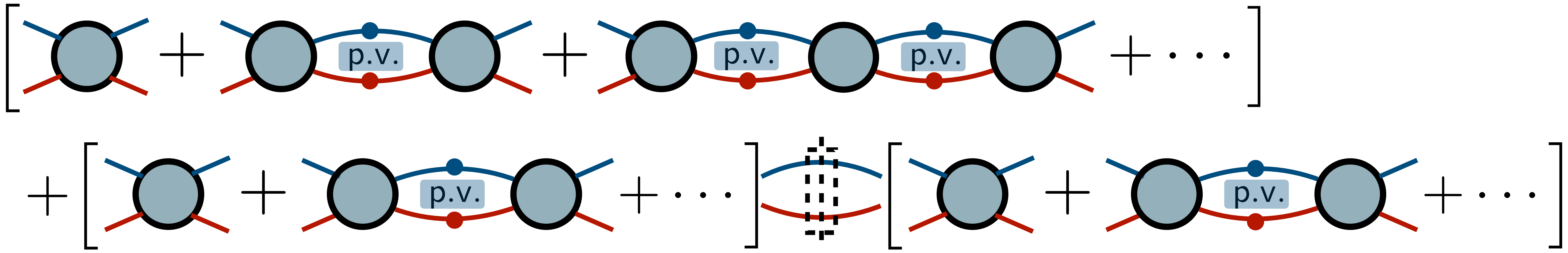

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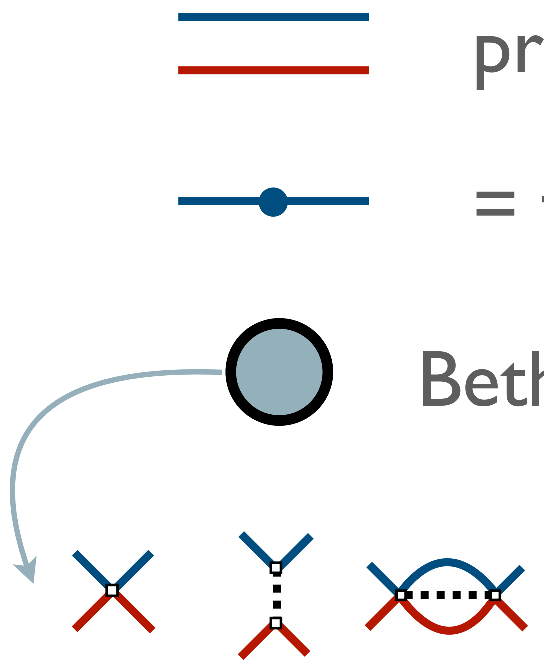

propagating hadrons

= fully dressed

Bethe-Salpeter kernel

$$= \sum_{\boldsymbol{k}} [\text{real, analytic}]$$

$$= \int_{\boldsymbol{k}} [\text{real, analytic}] + e^{-\mu L}$$

for $(2M_N)^2 < s < (2M_N + m_\pi)^2$

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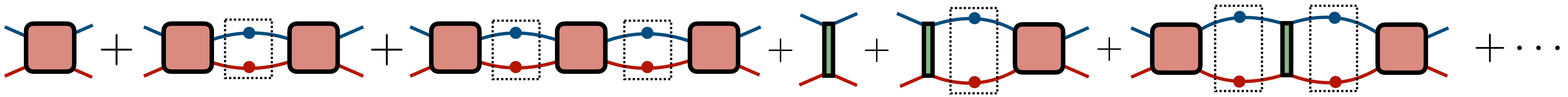
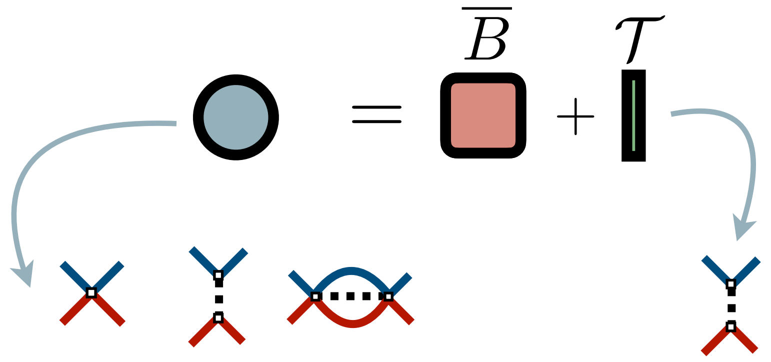
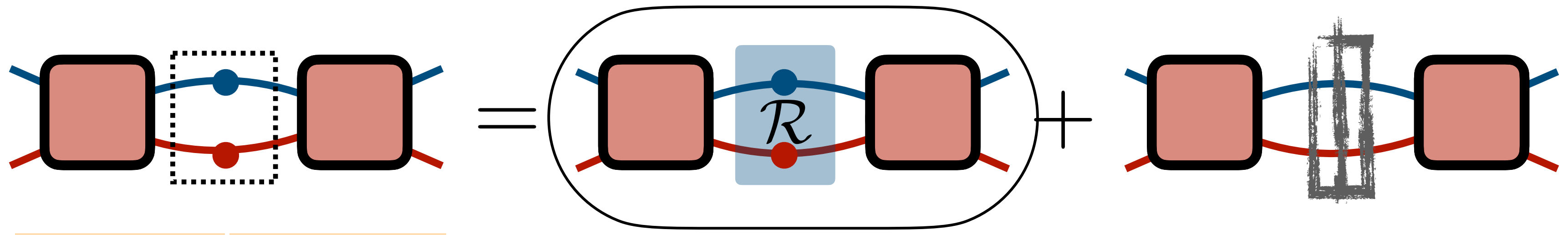
Some cutting details

$\mathcal{M}_L(P) =$ 





$$\frac{1}{L^3}\sum_{\boldsymbol{k}} \frac{1}{2\omega_{\boldsymbol{k}}} \frac{\mathcal{A}(\boldsymbol{k}_{\text{cm}}^2)\mathcal{B}(\boldsymbol{k}_{\text{cm}}^2)}{\boldsymbol{k}_{\text{cm}}^2 - p(E_{\text{cm}})^2}$$

$$\frac{1}{L^3}\sum_{\boldsymbol{k}} \frac{1}{2\omega_{\boldsymbol{k}}} \frac{\mathcal{A}(\boldsymbol{k}_{\text{cm}}^2)\mathcal{B}(p(E_{\text{cm}})^2)}{\boldsymbol{k}_{\text{cm}}^2 - p(E_{\text{cm}})^2} H(\boldsymbol{k}_{\text{cm}})$$

smooth cutoff H

abbreviating slightly we write…

$$\sum \frac{\mathcal{A}_{\text{off}}\mathcal{B}_{\text{off}}}{\boldsymbol{k}_{\text{cm}}^2 - p^2}$$

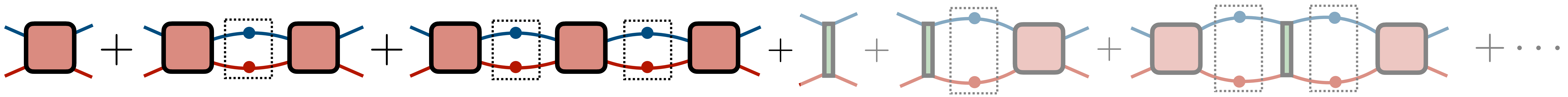
$$\sum \frac{\mathcal{A}_{\text{off}}\mathcal{B}_{\text{on}}}{\boldsymbol{k}_{\text{cm}}^2 - p^2}(1-H) + \sum \frac{\mathcal{A}_{\text{off}}\mathcal{B}_{\text{off}} - \mathcal{A}_{\text{off}}\mathcal{B}_{\text{on}}}{\boldsymbol{k}_{\text{cm}}^2 - p^2}$$

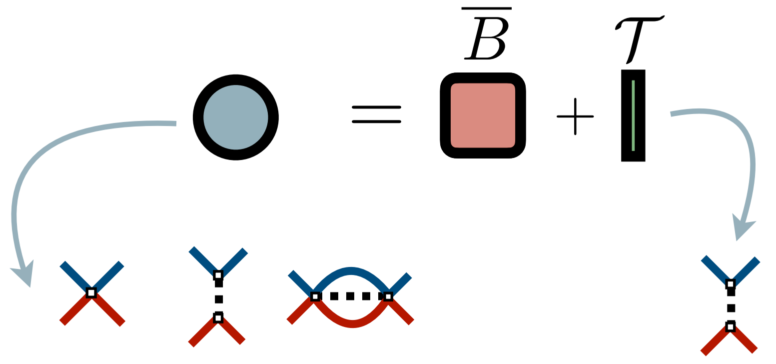
$$\sum \frac{\mathcal{A}_{\text{off}}\mathcal{B}_{\text{on}}}{\boldsymbol{k}_{\text{cm}}^2 - p^2} H$$

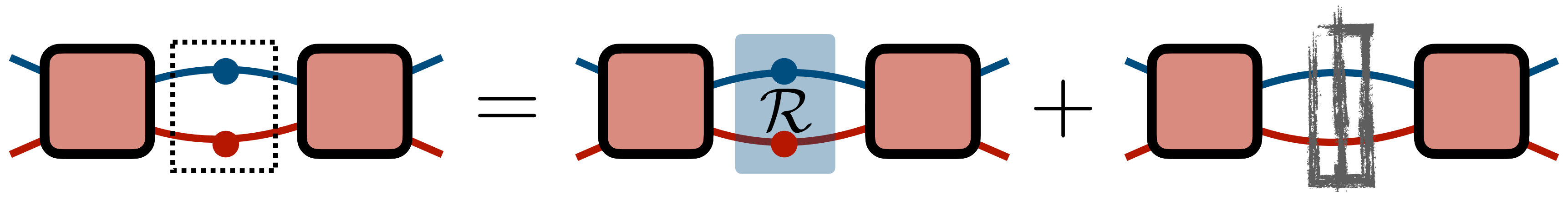
- ❑ We are still angular-momentum projecting everywhere *(not really the issue)*
- ❑ Sum is promoted to an additional index… can always be collapsed later
- ❑ Only the known log function is evaluated off-shell

$$-S(P,L)_{|\boldsymbol{k}'_{\text{cm}}|,\ell',m'\,,\;|\boldsymbol{k}_{\text{cm}}|,\ell,m}$$

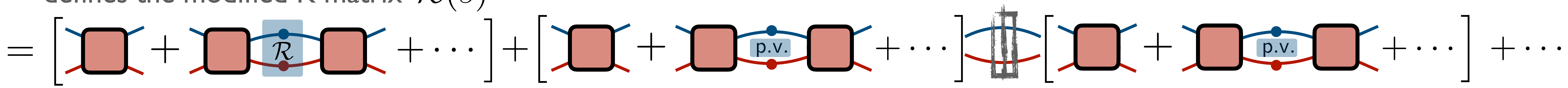
First sum the safe blobs

$\mathcal{M}_L(P) =$ 





defines the modified K matrix $\overline{\mathcal{K}}(s)$



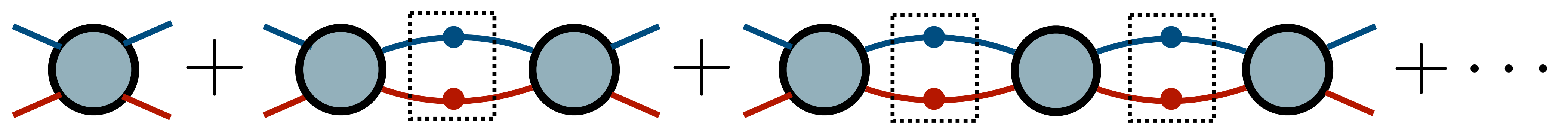
$-S(P,L)_{|\boldsymbol{k}'_{\text{cm}}|,\ell',m'\,,\ |\boldsymbol{k}_{\text{cm}}|,\ell,m}$

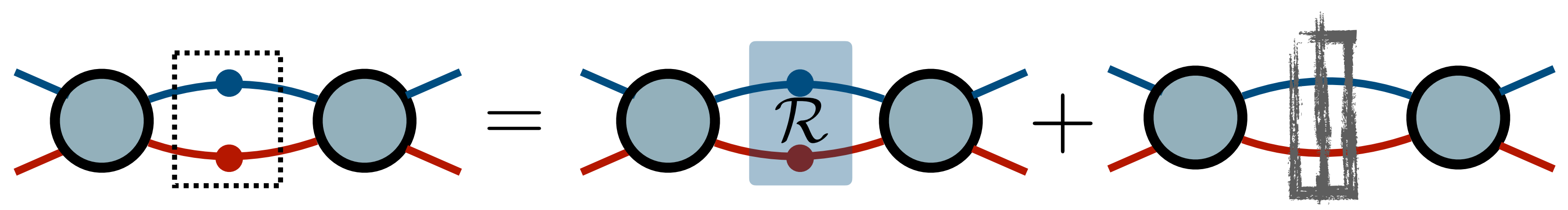
$$= \overline{\mathcal{K}}(s) - \overline{\mathcal{K}}(s)S(P,L)\overline{\mathcal{K}}(s) + \cdots = \frac{1}{\overline{\mathcal{K}}(s)^{-1} + S(P,L)}$$

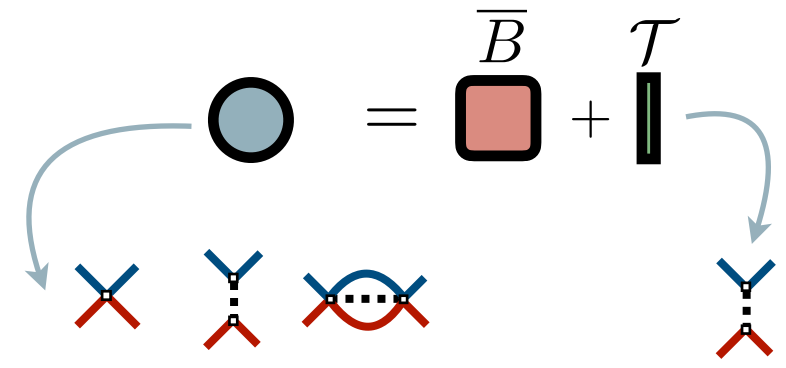
Key details:

- matrices on angular momentum $\otimes\ |\boldsymbol{k}_{\text{cm}}|$
- $\overline{K}(s)$ on shell but missing parts of diagrams
- but now have significant redefinition freedom

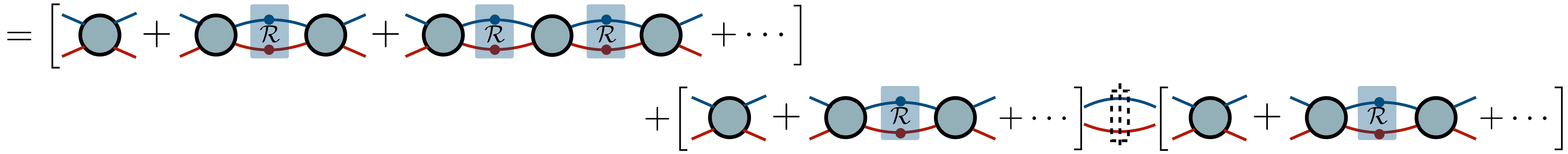
Completing the story…

$\mathcal{M}_L(P) =$ 



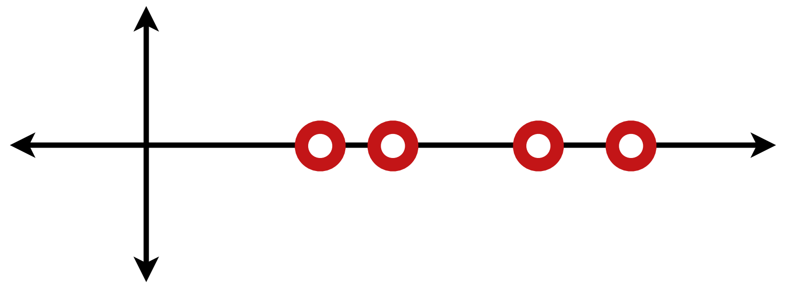


$-S(P,L)_{|\boldsymbol{k}'_{\mathrm{cm}}|,\ell',m'\;,\;|\boldsymbol{k}_{\mathrm{cm}}|,\ell,m}$

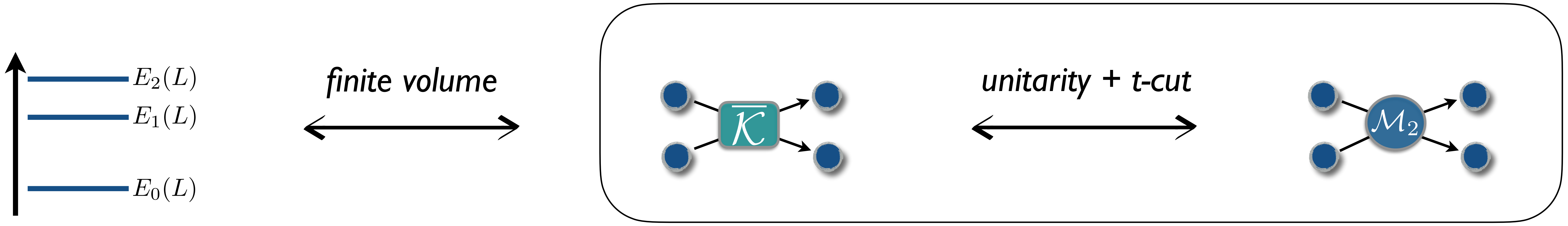


$-S(P,L)_{|\boldsymbol{k}'_{\mathrm{cm}}|,\ell',m'\;,\;|\boldsymbol{k}_{\mathrm{cm}}|,\ell,m}$

$$= [\overline{\mathcal{K}}(s) + g^2\mathcal{T}\,] - [\overline{\mathcal{K}}(s) + g^2\mathcal{T}\,]S(P,L)[\overline{\mathcal{K}}(s) + g^2\mathcal{T}\,] + \cdots = \frac{1}{[\overline{\mathcal{K}}(s) + g^2\mathcal{T}\,]^{-1} + S(P,L)}$$



Integral equations



- ☐ Exactly in the spirit of the three-particle approach

  - MTH, Sharpe (2015)
  - Agadjanov *et.al,* (2016) (Optical potential)
  -

- ☐ Define a finite-volume amplitude with the correct limit:

$$\mathcal{M}_L(P) = \frac{1}{[\overline{\mathcal{K}}(s) + g^2 \mathcal{T}\,]^{-1} + S(P,L)}$$

- ☐ Formally take an infinite-volume limit to derive an integral equation

$$\mathcal{M}(s,t) = \lim_{\text{os}} \lim_{\epsilon \to 0} \lim_{L \to \infty} \mathcal{M}_L(P + i\epsilon)_{|\boldsymbol{k}'_{\text{cm}}|\ell m, |\boldsymbol{k}_{\text{cm}}|\ell m}$$

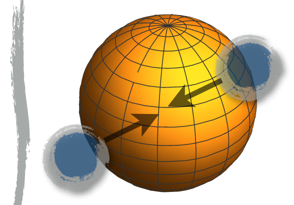
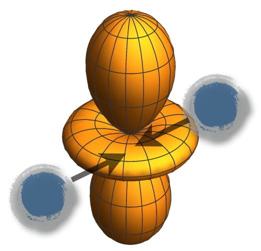
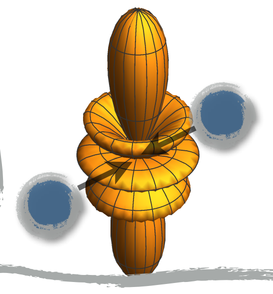
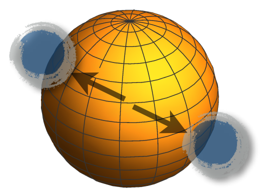
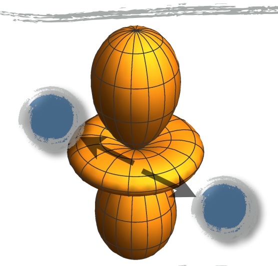
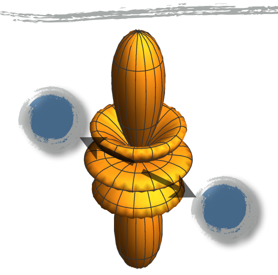
Note: *derivation strategy = logically separate from evaluation strategy*

QCD Fock space

- At low-energies QCD = hadronic degrees of freedom $\pi \sim \bar{u}d,\ K \sim \bar{s}u,\ p \sim uud$
- Overlaps of multi-hadron ***asymptotic states*** → S matrix

$|\pi\pi, \mathrm{in}\rangle$

$S(s) \equiv \langle \pi\pi, \mathrm{out}|$

			
	$e^{2i\delta_0(s)}$	0	0
	0	$e^{2i\delta_1(s)}$	0
	0	0	$e^{2i\delta_2(s)}$

depends on $s = E_{\mathrm{cm}}^2$ and angular variables

diagonal in angular momentum

$$\mathcal{M}_\ell(s) \propto e^{2i\delta_\ell(s)} - 1$$

angular momentum also plays an important role in resonant analysis

- ***An enormous space of information*** $|\pi\pi\pi\pi, \mathrm{in}\rangle \quad |K\overline{K}, \mathrm{in}\rangle \quad \cdots$