Computing scattering amplitude on the lattice involving Goldstone bosons

Ferenc Pittler

August 26, 2024

Outline

- Introduction:
  - Observables in continuum infinite volume scattering
  - Observables in finite volume lattice theory
  - Connection between the former two
- $2 \rightarrow 2$ scattering
  - meson-meson
  - meson-baryon
- $1 \rightarrow 2$ transition
- $3 \rightarrow 3$ scattering

The S matrix: continuum infinite volume

- Degrees of freedom in QCD (low T): hadrons: pion, kaon, proton
- Goal is to describe excited states composed by multi-hadrons.
- S matrix describes the overlap between multi-hadron states
- $S \cdot S^{\dagger} = 1$ unitary
- Describes the scattering between different channels
- Diagonal in angular momentum space

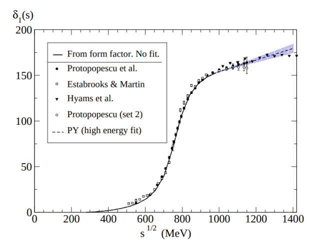
Scattering amplitude

$$\mathcal{M}(s) \propto e^{2i\delta_{\ell}(s)} - 1$$

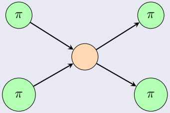
$\delta_{\ell}(s)$ phase shift

Resonances

Rho resonance $I = 1$ $\pi\pi$ scattering

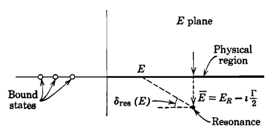


Pelaez Phys.Rev. D (2005)



- Cross-section $\propto |\mathscr{M}(s)|^2$
- Naive definition: Inflection point in the phase shift

Proper way of defining the resonance (Taylor (1972))



- Analytic continuation of $\mathscr{M}$ itself
- Poles below threshold at the real axis: Bound states
- Poles above threshold in the complex plane: Resonances

How the infinite-volume spectrum looks like?

Analyticity of $\mathscr{M}$

Optical theorem

$$\rho(s)|\mathscr{M}_\ell(s)|^2 = \mathrm{Im}\mathscr{M}_\ell(s)$$

Unique Solution

$$\mathscr{M}_\ell(s) = \frac{1}{\mathscr{K}^{-1}(s) - i\rho(s)}$$

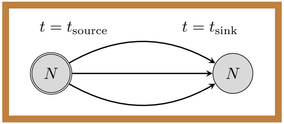
$$\rho(s) = \sqrt{1 - \frac{s}{4m^2}}, \text{ phase space}$$

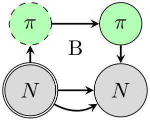
Conclusion

- The scattering amplitude has a square root branch cut above threshold
- Continuum of energy levels above threshold

Lattice field theory: Discrete spectrum

- We measure correlation functions
- $C(t) = \langle \mathcal{O}(t)\bar{\mathcal{O}}(0)\rangle = \sum_n e^{-E_n t} |\langle \Omega|\mathcal{O}(0)|n\rangle|^2$
- Observable energy values, matrix elements
- Assymptotic states $t \to \infty$
- To extract a physical quantity $L \to \infty, a \to 0, m \to m_{phys}$





Differences

- Discrete spectrum
- No branch cuts, no extra sheets
- No resonance poles

Finite volume lattice spectrum

Procedure:

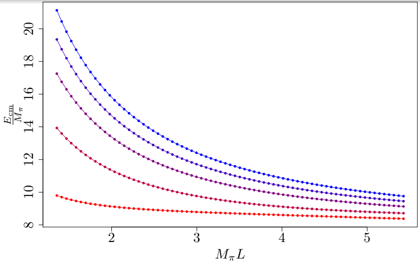
- Select a set of interpolating field $\mathcal{O}(i)$ with correct quantum numbers
- Compute set of correlation functions
- Finite volume ℓ is not a good quantum number: project to the relevant irreps
- Diagonalize the correlation matrices in the space of interpolating fields

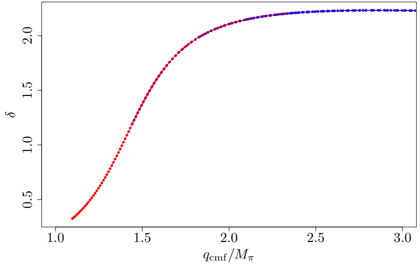
$$C_{ij}(t)v_j^n = \lambda^n(t,t_0)C_{ij}(t_0)v_j^n,\ \lambda^n(t,t_0) = e^{-E_n(t-t_0)}$$

- Extract the relevant energy value by fitting λ^n

Lüscher Method (Nucl.Phys.B 1991)

Connects finite-volume two particle energy spectra with infinite volume scattering amplitude



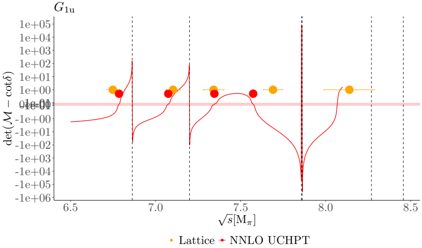


- Turn the finite volume to an advantage
- $\det(\mathcal{K}^{-1}(E_{cm}) + F(E_{cm})) = 0$
- F is a known function, $\mathcal{K}$ is scattering K matrix
- Example: We determined the spectrum ($E_{cm}/M_\pi(L)$)
- For each $E_{cm}/M_\pi(L)$ we determine $\delta_{\ell=1}(M_R, \Gamma_R)$
- Express $\mathcal{K}$ in terms of phase shifts: Points are collapsing on single phase shift curve
- Assume $\delta_{\ell=1}(s) = f(M_R, \Gamma_R, s)$, M_R, Γ_R can be estimated

Parametrizing the scattering amplitude

$\mathcal{K}$ matrix parametrizations

- Effective range
- Breit Wigner form
- Inverse amplitude method
- Chiral effective models



Energy level fit

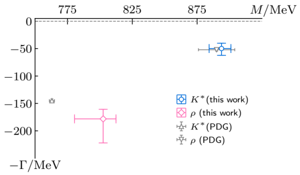
$$\sum_{n,m,\Lambda,\Lambda'} \left(E^L_{n,\Lambda} - E_{\text{Luescher}}(L,n,\Lambda)\right) \text{cov}^{-1}(n,m,\Lambda,\Lambda') \left(E^L_{m,\Lambda'} - E_{\text{Luescher}}(L,m,\Lambda')\right)$$

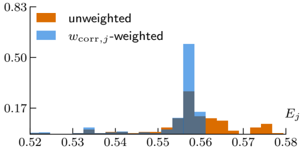
Meson-Meson scattering

$\pi\pi\ I = 1, \pi - K\ I = 1/2$, [$arXiv$: 2406.19193, 2406.19194] : *P.Boyle et al.*

RBC-UKQCD collaboration

- Physical point $2 + 1$ *flavor*
- $\rho(770), K^{\star}(892)$
- Fitrange systematics
- Different parametrization of the scattering amplitude
- Applications : transition matrix elements, continuum limit



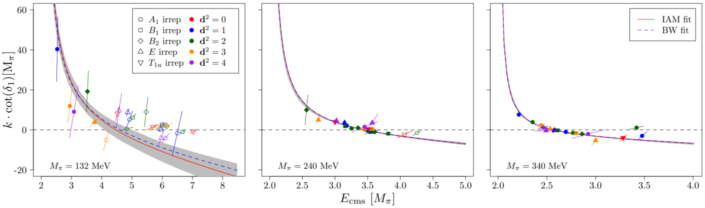


$\pi\pi$ scattering with inverse amplitude method

$\pi - \pi, I = 1, \rho$ resonance

ETM collaboration (Phys.Lett.B(2021))

- Fitting several masses together (including the physical point)
- Fitting the single finite volume pion masses and decay constants
- Avoid the need for scale setting in the fit
- Express amplitudes in terms of M_π / f_0



Coupled channel scattering

Mixing of partial waves

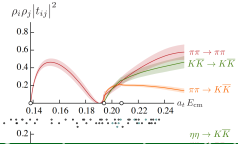
- Due to the cubic lattice different partial wave are mixing
- πN, πK scattering at $\vec{P} \neq 0$.

$$\det\left[\begin{pmatrix} \mathcal{K}_s^{-1} & 0 \\ 0 & \mathcal{K}_p^{-1} \end{pmatrix} + \begin{pmatrix} F_{ss} & F_{sp} \\ F_{ps} & F_{pp} \end{pmatrix}\right] = 0$$

Mixing of different channels

- Example $\bar{K}N, \pi\Sigma; \pi\pi, \bar{K}K$

$$\det\left[\begin{pmatrix} \mathcal{K}^{-1}_{\bar{K}N,\bar{K}N} & \mathcal{K}^{-1}_{\bar{K}N,\pi\Sigma} \\ \mathcal{K}^{-1}_{\pi\Sigma,\bar{K}N} & \mathcal{K}^{-1}_{\pi\Sigma,\pi\Sigma} \end{pmatrix} + \begin{pmatrix} F_{\bar{K}N,\bar{K}N} & 0 \\ 0 & F_{\pi\Sigma,\pi\Sigma} \end{pmatrix}\right] = 0$$



- $\sigma, f_0 \to \pi\pi, K\bar{K}, \eta\eta$
- Briceño et al., Phys.Rev. D 97 (2018) 5, 054513

Meson baryon scattering, Δ resonance

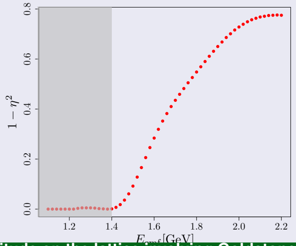
Problems

- Low $N\pi\pi$ threshold
- Signal-to-noise problem

$N\pi\pi$ threshold is very low

Data from http://gwdac.phys.gwu.edu

- Lüscher formula is valid only upto $N\pi\pi$ three-particle threshold



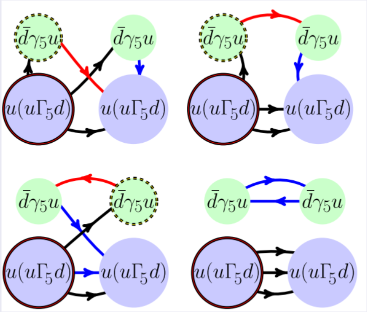
Meson baryon scattering, Δ resonance

Problems

- Low $N\pi\pi$ threshold
- Signal-to-noise problem

- Two correlated spatial sum at sink (pion, nucleon)
- Estimate it stochastically $D(x_N, x_\pi) = \sum_r \xi_r(N)\phi_r^\dagger(\pi)$
- Cut the diagram into factors
- Factors be combined to diagrams
- Many different diagrams share the same factors

$\pi N - \pi N$ diagrams



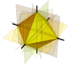
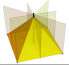
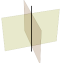
Correlation matrices

- irrep, irrep row(μ), # occurrences,# combinations of momenta
- As an example we have a 12×12 correlation matrix for the single hadron delta
- In the process of projection this matrix will be block diagonalized Gramm-Schmidt transformations

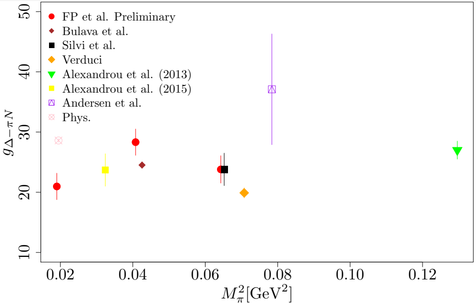
- Pion nucleon correlation matrix

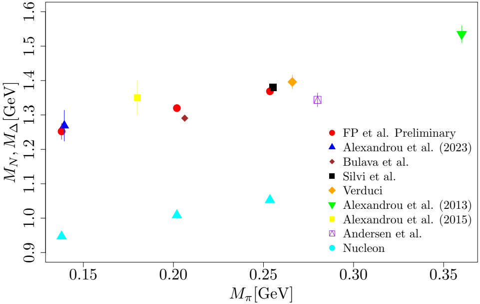
$\vec{p}_{\mathrm{tot}}$, irrep name	N_{dim}
$\vec{p} = (0,0,0), \mathrm{G1_u}$	8x8
$\vec{p} = (0,0,0), \mathrm{Hg}$	9x9
$\vec{p} = (0,0,1), \mathrm{G1}$	24x24
$\vec{p} = (0,0,1), \mathrm{G2}$	18x18
$\vec{p} = (1,1,0), \mathrm{(2)G}$	30x30
$\vec{p} = (1,1,1), \mathrm{(3)G}$	16x16
$\vec{p} = (1,1,1), \mathrm{F1}$	6x6
$\vec{p} = (1,1,1), \mathrm{F2}$	6x6

- Finite volume we no longer have continuous rotational symmetry
- Each irrep contain infinitely many continuum spin J

$\frac{L}{2\pi}\vec{P}$	$(0,0,0)$	$(0,0,1)$	$(0,1,1)$	$(1,1,1)$
Group LG	$O_h^{(D)}$	$C_{4v}^{(D)}$	$C_{2v}^{(D)}$	$C_{3v}^{(D)}$
Axis and planes of symmetry				
g_{LG}	96	16	8	12
$\Lambda(J^P) : \pi(0^-)$	$A_{1u}(0^-, 4^-, ...)$	$A_2(0, 1, ...)$	$A_2(0, 1, ...)$	$A_2(0, 1, ...)$
$\Lambda(J^P) : N(\frac{1}{2}^+)$	$G_{1g}(\frac{1}{2}^+, \frac{7}{2}^+, ...)$	$G_1(\frac{1}{2}, \frac{3}{2}, ...)$	$G(\frac{1}{2}, \frac{3}{2}, ...)$	$G(\frac{1}{2}, \frac{3}{2}, ...)$
$\Lambda(J^P) : \Delta(\frac{3}{2}^+)$	$H_g(\frac{3}{2}^+, \frac{5}{2}^+, ...)$	$G_1(\frac{1}{2}, \frac{3}{2}, ...) \oplus G_2(\frac{3}{2}, \frac{5}{2}, ...)$	$(2)G(\frac{1}{2}, \frac{3}{2}, ...)$	$G(\frac{1}{2}, \frac{3}{2}, ...) \oplus F_1(\frac{3}{2}, \frac{5}{2}, ...) \oplus F_2(\frac{3}{2}, \frac{5}{2}, ...)$

Meson Baryon scattering: Δ resonance





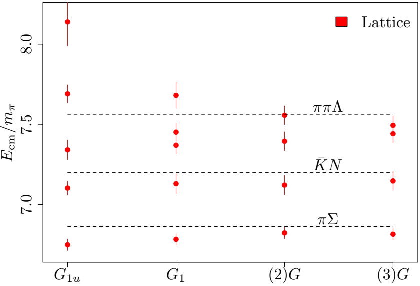
Δ-resonance

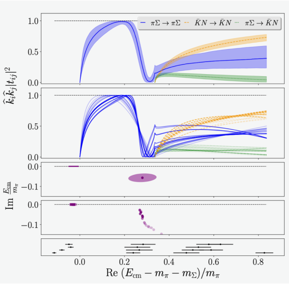
- Lightest baryon resonance
- Breit-Wigner form for the scattering amplitude
- $\ell = 1 \rightarrow \tan\delta_{\ell=1} = \frac{\sqrt{s}\Gamma(\Gamma_R, s)}{{M_R}^2 - s}$
- $\Gamma(\Gamma_R, s) = \Gamma_R \left(\frac{q}{q({M_R}^2)}\right)^3 \frac{{M_R}^2}{s}$
- The coupling obtained by analytic-continuation to the resonance pole
- First step towards $\langle N | \mathcal{J} | \Delta \rangle$

$\Lambda(1405)$: $\bar{K}N, \pi\Sigma$ coupled channel analysis

Bulava et al. (PRL 2024)

- $D200$ CLS ensemble: $N_f = 2+1$ non perturbatively improved Wilson
- $m_\pi \sim 200\text{MeV}, \ m_\pi L \sim 4.18$
- Virtual bound state below the $\pi\Sigma$ threshold
- Resonance below the $\bar{K}N$ threshold





Work in progress: Maxim Mai, FP

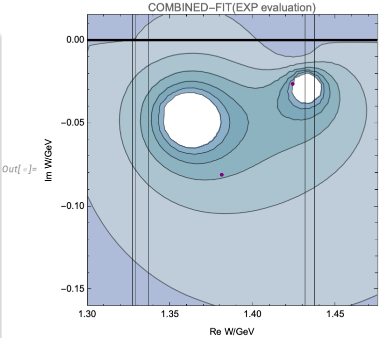
Potential model

- Degrees of freedom: Meson and Baryon octet
- We compute partial wave amplitudes $f_{0+}(E_2)$
- Scattering length: $f_{0+}^{MB}(m_M + m_B) = a_{MB}$

$$T(E_2) = 8\pi E_2 f_{0+} = -V(E_2)\frac{1}{1 - G(E_2)V(E_2)}$$

$$V_{ij}^{\mathrm{WT}}(\sqrt{s}) = -\frac{C_{ij}^{\mathrm{WT}}}{8F_iF_j}\mathcal{N}_i\mathcal{N}_j(2\sqrt{s} - m_i - m_j)$$

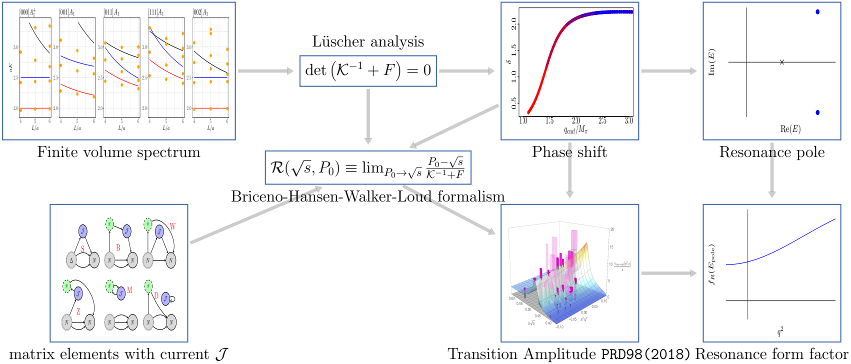
$$V_{ij}^{\mathrm{NLO}}(\sqrt{s}) = \frac{\mathcal{N}_i\mathcal{N}_j}{F_iF_j}\left(C_{ij}^{\mathrm{NLO1}} - 2C_{ij}^{\mathrm{NLO2}}\left(E_iE_j + \frac{q_i^2q_j^2}{3\mathcal{N}_i\mathcal{N}_j}\right)\right).$$



- Fitting lattice + experimental data
- Pole structure at the physical point

Transition amplitude

Formalism available for $1 + \mathcal{J} \to 2$ and for $2 + \mathcal{J} \to 2$



$1 \to 2$ Hansen, Sharpe (2012) • Briceño, Hansen, Walker-Loud (2015) • Briceño, Hansen (2016)

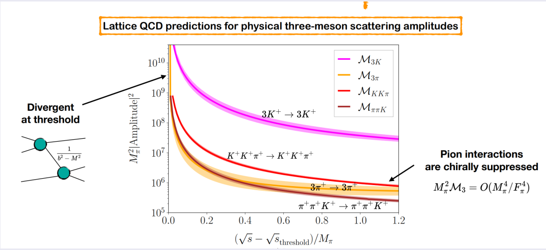
$2 \to 2$ Briceño, Hansen (2017), Baroni, Briceño, Hansen, Ortega-Gama (2018), Briceño, Hansen, Jackura (2020)

Three particle scattering

The formalism:

- 2 particle spectrum: Fit: $\det_{\ell,m}\left[\mathcal{K}_2^{-1}+F_2\right]=0$
- 3 particle spectrum: Fit $\det_{k,\ell,m}\left[\mathcal{K}_{df,3}^{-1}+F_3\right]=0$
- $\mathcal{K}_2, \mathcal{K}_{df,3} \to \mathcal{M}_2, \mathcal{M}_3$ unitarity, integral equations

Fernando Lopez et al. [Lattice 24]



Summary,outlook

- 2-2 First calculations at the physical point
- 3-3 also at the physical point
- Using the scattering amplitude description logical next-step would be computing resonance form factors
- Taking correct Continuum limit
- Taking into account correctly left-hand cuts

Thank you for your attention

Support is acknowledged from the project EXCELLENCE/0421/0195 “Nice quarks,” cofinanced by the European Regional Development Fund and the Republic of Cyprus through the Research and Innovation Foundation

Με τη συγχρηματοδότηση της Ευρωπαϊκής Ένωσης

Κυπριακή Δημοκρατία

ΙΔΡΥΜΑ ΕΡΕΥΝΑΣ ΚΑΙ ΚΑΙΝΟΤΟΜΙΑΣ